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DIRECTIONAL DIFFERENTIABILITY OF THE OPTIMAL VALUE IN QUADRATIC PROGRAMMING PROBLEMS UNDER LINEAR CONSTRAINTS ON HILBERT SPACES

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ABSTRACT

We investigate the first-order directional differentiability of the optimal value function in parametric quadratic programming problems under linear constraints in Hilbert spaces. We derive an explicit formula for computing the directional derivative of the optimal value function in cases where the quadratic part of the objective function is in Legendre form.

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TÍNH KHẢ VI THEO HƯỚNG CỦA HÀM GIÁ TRỊ TỐI ƯU TRONG BÀI TOÁN QUY HOẠCH TOÀN PHƯƠNG VỚI RÀNG BUỘC TUYẾN TÍNH TRÊN KHÔNG GIAN HILBERT

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Quy hoạch toàn phương, không gian Hilbert, dạng Legendre, tập nghiệm, hàm giá trị tối ưu, khả vi theo hướng.

TÓM TẮT

Chúng ta quan tâm đến tính khả vi cấp một theo hướng của hàm giá trị tối ưu trong bài toán quy hoạch toàn phương tham số hóa với ràng buộc tuyến tính trong không gian Hilbert. Chúng ta suy ra một công thức tường minh để tính đạo hàm theo hướng cấp một của hàm giá trị tối ưu trong trường hợp dạng toàn phương trong hàm mục tiêu là dạng Legendre.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm denoted by $\| \cdot \|$. Let $\mathcal{L}(\mathcal{H})$ be the space of continuous linear operators from $\mathcal{H}$ into $\mathcal{H}$ equipped with the operator norm induced by the vector norm in $\mathcal{H}$ and also denoted by $\| \cdot \|$. The norm in the product space $X_1 \times \ldots \times X_k$ of the normed spaces $X_1, \ldots, X_k$ is defined by $\|(x_1, \ldots, x_k)\| = \max\{\|x_1\|, \ldots, \|x_k\|\}$. Let

$$\Omega := \mathcal{L}(\mathcal{H}) \times \mathcal{H}^{m+1} \times \mathbb{R}^m.$$

We consider parameterized quadratic programming problem of the form

$$\begin{cases} \min \ f(x,\omega) := \frac{1}{2}\langle x, T_0 x\rangle + \langle c_0, x\rangle \\ \text{s. t. } x \in \mathcal{H} : g_i(x,\omega) := \langle c_i, x\rangle + \alpha_i \leqslant 0, \ i = 1, \ldots, m, \end{cases} \qquad (QP_\omega)$$

depending on the parameter vector

$$\omega = (T_0, c_0, c_1, \ldots, c_m, \alpha_1, \ldots, \alpha_m) \in \Omega,$$

where $T_0 : \mathcal{H} \to \mathcal{H}$ is a continuous linear self-adjoint operator, $c_0, c_i \in \mathcal{H}$, and α_i are real numbers, $i = 1, 2, \ldots, m$.

Put $\lambda = (\lambda_1, \ldots, \lambda_m)$ where $\lambda_i, i = 1, \ldots, m$ are real numbers. By

$$F(x,\omega) := \{x \in \mathcal{H} : g_i(x,\omega) := \langle c_i, x\rangle + \alpha_i \leqslant 0, \ i = 1, \ldots, m\}$$

and

$$L(x, \omega, \lambda) := f(x, \omega) + \sum_{i=1}^{m} \lambda_i g_i(x, \omega),$$

we denote *constraint set* and the *Lagrangian function* of (QP_ω), respectively.

For a given point ω_0 in the parameter space Ω, we view the corresponding problem (QP_{ω_0}) as an unperturbed problem, and investigate differentiability properties of the *optimal value function*

$$\varphi : \Omega \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$\varphi(\omega) = \begin{cases} \inf\{f(x, \omega) : x \in F(\omega)\} & \text{if } F(\omega) \neq \emptyset \\ +\infty & \text{if } F(\omega) = \emptyset. \end{cases}$$

The solution set of (QP_ω) defined by

$$\mathrm{Sol}(QP_\omega) = \{x \in F(x, \omega) \mid f(x, \omega) = \varphi(\omega)\}.$$

Stability is an important topic in optimization theory and practical applications. Directional differentiability of the optimal value function for QP problems in Euclidean spaces has been investigated extensively in various versions; see [10] and the references therein. The continuity of the solution set mappings and of the optimal value function for parametric quadratic programming problems in a Hilbert space have been intensively studied in [5]. Optimality conditions for quadratic programming problems in Hilbert spaces have been intensively studied in [4]. Various aspects of the value function in optimization have been studied in [2, 6, 7, 8, 9] and the references therein. Since quadratic programming is a class of optimization problems, the results in optimization can be applied to convex and nonconvex quadratic programming problems.

This paper studies parametric quadratic programming problems in a Hilbert space. The main results of the paper concern differentiability properties of the optimal value function of the problem whose quadratic part of the objective function is a Legendre form. We would like to stress that the notion of Legendre form, which originated in the Calculus of Variations is crucial for the solution existence theorem of quadratic programming problems in Hilbert spaces. In [3], Dong and Tam constructed an example to show that the conclusion of that theorem fails if the assumption on the Legendre property of the quadratic form is omitted.

The remainder of the paper is organized as follows. In Section 3 we study differentiability properties of the optimal value function of the problem (QP_ω) where feasible set is also subject to perturbations. Concluding remarks are offered in Section 3.

2. Lemmas

Throughout this paper, we denote $Dg(x) : X \to Y$ as the Gâteaux derivative of the mapping $g : X \to Y$ at the point $x \in X$ and $D_x g(x, \omega)$ as the partial derivative of the mapping g :

$X \times \Omega \to Y$. It is said that g is Gateaux differentiable at x if g is directionally differentiable at x and the directional derivative $g'(x, h)$ is linear and continuous in h. That is. $g(x, \cdot) : X \to Y$ is a continuous linear operator. We denote this operator (when it exists) by $Dg(x)$, i.e., $Dg(x)h = g'(x, h)$.

Let $x_0 \in \text{Sol}(QP_{\omega_0})$. For a given direction $d \in \Omega$ consider the following linearization of (QP_{ω_0})

$$\begin{cases} \min \ D_x f(x_0, \omega_0)h + f(x_0, d) \\ \text{s. t. } h \in \mathcal{H} : D_x g_i(x_0, \omega_0)h + g_i(x_0, d) \leqslant 0, \ i \in I(x_0, \omega_0), \end{cases} \tag{PL_d}$$

where $D_x f(x_0, \omega_0)h = \langle T_0 x_0 + c, h\rangle$, $D_x g_i(x_0, \omega_0)h = \langle c_i, h\rangle$ and $I(x_0, \omega_0)$ denotes the index set of active at x_0 inequality constraints, i.e.,

$$I(x_0, \omega_0) = \{i \mid g_i(x_0, \omega_0) = 0, i = 1, 2, \dots, m\}.$$

Let X and Y be Banach spaces and K is a closed convex subset of Y. Consider the sets defined by abstract constraints in the form

$$\Phi(\omega) := \{x \in X \mid G(x, \omega) \in K\}$$

where $\omega \in \Omega$, Ω is a topological space and $G : X \times \Omega \to Y$ is a continuous mapping.

We say that Robinson's constraint qualification holds at a point $x_0 \in X$ such that $G(x_0.\omega) \in K$, with respect to the mapping $G(\cdot, \omega_0)$ and the set K, if the following regularity condition is satisfied.

$$0 \in \text{int}\{G(x_0, \omega_0) + D_x G(x_0, \omega_0)X - K\}.$$

Lemma 2.1. *Let $x_0 \in F(x, \omega_0)$ and suppose that there exists $\bar{x} \in \mathcal{H}$ such that $g_i(\bar{x}) < 0$ for all $i = 1, \dots, m$ (Slater condition). Then the following holds*

a) *Robinson's constraint qualification is satisfied at x_0.*

b) *The constraint set of the problem (PL_d) is nonempty.*

Proof. It is clear that g_i are continuous and convex. By convexity of g_i, we have

$$g_i(x_0 + t(\bar{x} - x_0)) = g_i((1-t)x_0 + t\bar{x}) \leq (1-t)g_i(x_0) + tg_i(\bar{x}) \quad \forall t \in (0, 1). \tag{1}$$

We have

$$\langle c_i, \bar{x} - x_0\rangle = \lim_{t \downarrow 0} \frac{g_i(x_0 + t(\bar{x} - x_0)) - g_i(x_0)}{t} \tag{2}$$

Combining (1) with (2) we obtain

$$\langle c_i, \bar{x} - x_0\rangle \leq \lim_{t \downarrow 0} \frac{(1-t)g_i(x_0) + tg_i(\bar{x}) - g_i(x_0)}{t} = g_i(\bar{x}) - g_i(x_0)$$

Put $\bar{h} = \bar{x} - x_0$. It follows from the above inequality that

$$\langle c_i, \bar{h}\rangle + g_i(x_0) \leqslant g_i(\bar{x}) < 0. \tag{3}$$

a) Let $G : \mathcal{H} \times \Omega \to \mathbb{R}^m$ defined by

$$G(x,\omega) = (g_1(x,\omega), \dots, g_m(x,\omega)).$$

Then the constraint set of (QP_ω) can be rewritten as

$$F(x,\omega) = \{x \in \mathcal{H} : G(x,\omega) \in R^m_-\}$$

We have

$$DG(x_0,\omega_0)\bar{h} = \big(\langle c_1, \bar{h}\rangle, \langle c_2, \bar{h}\rangle, \dots, \langle +c_m, \bar{h}\rangle\big)$$

and hence (3) can be rewritten as

$$G(x_0,\omega_0) + DG(x_0,\omega_0)\bar{h} \in \operatorname{int}\{R^m_-\}.$$

Since R^m_- has a nonempty interior, by Lemma 2.99 in [1], Robinson's constraint qualification holds at x_0.

b) It follows from (3) that

$$D_x g_i(x_0,\omega_0)\bar{h} = \langle c_i, \bar{h}\rangle < 0, i \in I(x_0).$$

Hence, for each $i \in I(x_0)$, there exists $\beta_i > 0$ large enough such that

$$D_x g_i(x_0,\omega_0)(\beta_i \bar{h}) + g_i(x_0, d) \leqslant 0.$$

Let $\beta = \max\{\beta_i, i \in I(x_0)\}$. We have

$$D_x g_i(x_0,\omega_0)(\beta \bar{h}) + g_i(x_0, d) \leqslant 0 \ \ \forall i \in I(x_0).$$

This shows that constraint set of (PL_d) is nonempty. The proof is complete. $\square$

Dual of the above problem (PL_d) can be written as

$$\max_{\lambda \in \Lambda(x_0,\omega_0)} f(x_0, d) + \sum_{i=1}^{m} \lambda_i g_i(x_0, d).$$

It is not difficult to show that

$$f(x_0, d) + \sum_{i=1}^{m} \lambda_i g_i(x_0, d) = D_\omega L(x_0, \lambda, \omega_0)d.$$

Therefore, the dual of (PL_d) can be written in the form

$$\max_{\lambda \in \Lambda(x_0,\omega_0)} D_\omega L(x_0, \lambda, \omega_0)d, \tag{DL_d}$$

where $\Lambda(x_0,\omega_0)$ denotes the set of all Lagrange multipliers at that is, the set of $\lambda = (\lambda_1, \dots, \lambda_m) \in R^m_+$ such that

$$\begin{cases} T\bar{x} + c + \sum\limits_{i=1}^{m} \lambda_i c_i = 0 \\ g_i(x) \leqslant 0, \\ \lambda_i g_i(x) = 0, \\ i = 1, \dots, m. \end{cases} \tag{4}$$

Lemma 2.2. *Let $x_0 \in F(x, \omega_0)$ and suppose that there exists $\bar{x} \in \mathcal{H}$ such that $g_i(\bar{x}) < 0$ for all $i = 1, \dots, m$. Then for each h is a feasible point of the problem (PL_d) there exists $v \in \mathcal{H}$ satisfying*

$$\langle c_i, v\rangle + 2D_x g_i(x_0, \omega)h + \langle Th, h\rangle < 0, \text{ for all } i \in I_1(x_0, \omega_0), \tag{5}$$

where $I_1(x_0, \omega_0) = \{i \in I(x_0, \omega_0) \mid D_x g_i(x_0, \omega_0)h + g_i(x_0, d) = 0\}$.

Proof. We first prove that there exists $h_0 \in \mathcal{H}$ such that $\langle c_i, h_0\rangle + g_i(x_0, \omega_0) < 0$ for all $i \in I(x_0, \omega_0)$. Suppose that $g_i(\bar{x}) < 0$ for all $i = 1, \dots, m$. Then there exists $\bar{h} = \bar{x} - x_0 \in \mathcal{H}$ such that $\langle c_i, \bar{h}\rangle < 0$ for all $i \in I(x_0)$. Hence, for each $i \in I(x_0)$, there exists $\gamma_i > 0$ large enough such that

$$D_x g_i(x_0, \omega_0)(\gamma_i \bar{h}) + g_i(x_0, d) < 0.$$

Let $h_0 = \gamma \bar{h}$ where $\gamma = \max\{\gamma_i, i \in I(x_0)\}$. We have

$$D_x g_i(x_0, \omega_0)h_0 + g_i(x_0, d) < 0 \ \ \forall i \in I(x_0).$$

Chose $\alpha > 0$ is sufficiently large. Let $v = \alpha(h_0 - h)$. Then, for all $i \in I_1(x_0, \omega_0)$ we have

$$\begin{aligned}
&\langle c_i, v\rangle + 2D_x g_i(x_0, \omega)h + \langle Th, h\rangle = \\
&\quad = \alpha\langle c_i, (h_0 - h)\rangle + 2D_x g_i(x_0, \omega)h + \langle Th, h\rangle \\
&\quad = \alpha\big[\langle c_i, h_0\rangle + g_i(x_0, \omega_0)\big] + 2D_x g_i(x_0, \omega)h + \langle Th, h\rangle < 0.
\end{aligned}$$

The proof is complete. □

We shall denote by $\mathcal{D}(x_0, v)$ the set of v's satisfying (5).

Proposition 2.1. *Suppose that there exists $\bar{x} \in \mathcal{H}$ such that $g_i(\bar{x}) < 0$ for all $i = 1, \dots, m$. Then $\mathrm{val}(PL_d) = \mathrm{val}(DL_d) < +\infty$ and the common value $\mathrm{val}(PL_d) = \mathrm{val}(DL_d)$ is finite if and only if the set $\Lambda(x_0, \omega_0)$ of Lagrange multipliers is nonempty.*

Proof. Using the similar argument as in the proof of Lemma 2.2, we obtain that there exists $h_0 \in \mathcal{H}$ such that $D_x g_i(x_0, \omega_0)h_0 + g_i(x_0, d) < 0$ for all $i \in I(x_0)$ and hence Robinson's constraint qualification holds. It is not difficult to show that the objective function of (DL_d) is lower semicontinuous and the (DL_d) is a convex problem. Therefore, by [1, Propostion 4.21] we obtain that $\mathrm{val}(PL_d) = \mathrm{val}(DL_d) < +\infty$ and the common value $\mathrm{val}(PL_d) = \mathrm{val}(DL_d)$ is finite if and only if the set $\Lambda(x_0, \omega_0)$ of Lagrange multipliers is nonempty. □

3. Main results

In this paper, we consider the case that (QP_{ω_0}) certainly has a solution. For each (QP_ω), we consider the following problem

$$\min\{\frac{1}{2}\langle v, T_0 v\rangle : v \in 0^+F(x, \omega)\}. \tag{CRP}$$

where

$$0^+F(\omega) = \{v \in \mathcal{H} \mid \langle c_i, v\rangle \leqslant 0,\ \forall i = 1, ..., m\}.$$

Let us denote by Sol(CRP) the solution set of (CRP).

The following theorem will describe sufficient conditions for $\varphi(\cdot)$ to be first order directionally differentiable and give explicit formulas for computing this directional derivative of $\varphi(\cdot)$. For this, we will need the following assumption:

Assumption (A).

$$\liminf_{n\to\infty} \frac{\langle x_n - x_0, T_0(x_n - x_0)\rangle}{t_n} \geq 0 \tag{A}$$

The assumption (A) was introduced by Tam and in [10]. For a detailed discussion of this assumption, see [10].

In the following theorem we extend the result in [10] to Hilbert spaces.

Theorem 3.1. *Consider the problem* (QP_ω) *where* $\langle x, T_0x\rangle$ *is a Legendre form. Suppose that*

(i) *there exists* $\bar{x} \in \mathcal{H}$ *such that* $g_i(\bar{x}) < 0$ *for all* $i = 1, \dots, m$,

(ii) $\mathrm{Sol(CRP)} = \{0\}$,

(iii) *the assumption* (A) *is satisfied.*

Then, the optimal value function $\varphi(\cdot)$ *is Hadamard directionally differentiable at* ω_0 *in the direction* d, *and*

$$\varphi'(\omega_0, d) = \inf_{x\in \mathrm{Sol}(QP_{\omega_0})} \sup_{\lambda\in\Lambda(x_0,\omega_0)} D_\omega L(x_0, \lambda, \omega_0)d. \tag{6}$$

Proof. Since $\langle x, T_0x\rangle$ is a Legendre, and since $\mathrm{Sol(CRP)} = \{0\}$, by [5, Lemma 2] we deduce that $\mathrm{Sol}(QP_{\omega_0})$ is nonempty. Let $x_0 \in \mathrm{Sol}(QP_{\omega_0})$.

By Lemma 2.1 we obtain that, under the Slater condition, constraint set of (PL_d) is nonempty. Let h be a feasible point of the problem (PL_d). Consider a point $v \in \mathcal{D}(x_0, v)$ and let $x(t) = x_0 + th + \frac{t^2}{2}v$ be the corresponding parabolic sequence. For any $\omega(t) := \omega_0 + td$ we have

$$\begin{aligned} g_i(x(t), \omega(t)) = g_i(x_0, \omega_0) &+ t\Big[D_x g_i(x_0, \omega_0)h + g_i(x_0, d)\Big] \\ &+ \frac{t^2}{2}\Big[\langle c_i, v\rangle + 2D_x g_i(x_0, \omega)h + \langle Th, h\rangle\Big] + o(t^2) \end{aligned} \tag{7}$$

It is clear that if $i \notin I(x_0, \omega_0)$ then $g_i(x(t), \omega(t)) \leqslant 0$ for $t > 0$ small enough. For $i \in I(x_0, \omega_0)$, combining (7) with Lemma 2.2, we obtain that $g_i(x(t), \omega(t)) \leqslant 0$ for $t > 0$ small enough. Consequently, $x(t) \in F(x, \omega(t))$ for $t > 0$ small enough. It follows that

$$\begin{aligned} \varphi(\omega_0 + td) &\leqslant f(x_0 + th + \frac{t^2}{2}v, \omega(t)) \\ &= f(x_0, \omega_0) + t\big[D_x g_i(x_0, \omega_0)h + g_i(x_0, d)\big] + o(t). \end{aligned}$$

and hence, since $\varphi(\omega_0) = f(x_0, \omega_0)$,

$$\limsup_{t\downarrow 0} \frac{\varphi(\omega_0 + td) - \varphi(\omega_0)}{t} \leqslant D_x g_i(x_0, \omega_0)h + g_i(x_0, d)$$

Since h is an arbitrary feasible point of (PL_d) and by Proposition 2.1 , we obtain

$$\limsup_{t\downarrow 0} \frac{\varphi(\omega_0 + td) - \varphi(\omega_0)}{t} \leqslant \sup_{\lambda \in \Lambda(x_0,\omega_0)} D_\omega L(x_0, \lambda, \omega_0)d. \tag{8}$$

Consider a sequence $t_n \downarrow 0$ as $n \to \infty$, and let $\omega_n = \omega_0 + t_n d$. Since F is nonempty, $\langle x, T_0 x\rangle$ is a Legendre and Sol(CRP) $= \{0\}$, by [5, Lemma 2, Lemma 5, Lemma 6,] we deduce that Sol$(\omega_0 + t_n d)$ is bounded, nonempty for n large enough. Hence there exists $x_n \in$ Sol$(\omega_0 + t_n d)$. By Sol$(\omega_0 + t_n d)$ is bounded, $\{x_n\}$ it has a weakly convergent subsequence. Without loss of generality, we can assume that x_n itself weakly converges to some x_0. Let any $x \in F(x, \omega_0)$. By [5, Lemma 3], the set-valued map $\omega \mapsto F(x, \omega)$ is lower semi-continuous at ω. Thus, there exits $\{y_n\} \subset F(x, \omega_0 + t_n d)$ such that $y_n \to x$. Since $x_n \in$ Sol$(\omega_0 + t_n d)$, we have

$$g_i(x_n, \omega_0 + t_n d) \leqslant 0, \quad i = 1, 2, \ldots, m$$

and

$$f(x_n, \omega_0 + t_n d) \leqslant f(y_n, \omega_0 + t_n d).$$

Taking $\liminf$ in the both above inequalities as $k \to \infty$ we have $f(x_0, \omega_0) \leqslant f(x, \omega_0)$ and $g_i(x, \omega_0) \leq 0$. These imply $x_0 \in$ Sol(QP_{ω_0}).

We have

$$\varphi(\omega_0 + t_n d) - \varphi(\omega_0) = f(x_n, \omega_0 + t_n d) - f(x_0, \omega_0). \tag{9}$$

Take any $\lambda \in \Lambda(x_0, \omega_0)$. Since

$$\lambda_i g_i(x_0, \omega_0) = 0, \quad \lambda_i \geqslant 0 \quad \forall i = 1, 2, \ldots, m$$

and

$$g_i(x_n, \omega_0 + t_n d) \leqslant 0, \quad \forall i = 1, 2, \ldots, m$$

we get from (9)

$$\begin{aligned}
&\varphi(\omega_0 + t_n d) - \varphi(\omega_0) \\
&\quad \geqslant f(x_n, \omega_0 + t_n d) - f(x_0, \omega_0) - \sum_{i=1}^{m} \lambda_i g_i(x_0, \omega_0) + \sum_{i=1}^{m} \lambda_i g_i(x_n, \omega_0 + t_n d) \\
&\quad = f(x_n, \omega_0) + t_n f(x_k, \omega_0) - f(x_0, \omega_0) - \sum_{i=1}^{m} \lambda_i g_i(x_0, \omega_0) + \sum_{i=1}^{m} \lambda_i \big[g_i(x_n, \omega_0) + t_n g_i(x_n, d)\big] \\
&\quad = t_n \big[f(x_n, d) + \sum_{i=1}^{m} \lambda_i g_i(x_n, d)\big] + f(x_n, \omega_0) - f(x_0, \omega_0) + \sum_{i=1}^{m} \lambda_i \big[g_i(x_n, \omega_0 - g_i(x_0, \omega_0)\big] \\
&\quad = t_n D_\omega L(x_0, \lambda, \omega_0)d + \frac{1}{2}\langle x_n - x_0, T_0(x_n - x_0)\rangle \\
&\qquad + \langle Tx_0 + c + \sum_{i=1}^{m} c_i, x_n - x_0\rangle.
\end{aligned}$$

Since $\lambda \in \Lambda(x_0, \omega_0)$, by [4, Theorem 3.2], $Tx_0 + c + \sum_{i=1}^{m} c_i = 0$. Hence, we have

$$\varphi(\omega_0 + t_n d) - \varphi(\omega_0) \geqslant t_n D_\omega L(x_0, \lambda, \omega_0) d + \frac{1}{2}\langle x_n - x_0, T_0(x_n - x_0)\rangle$$

Multiplying both sides of this equality by t_n^{-1}, take lim inf as $n \to \infty$ and using assumption (A), we obtain

$$\liminf_{t \downarrow 0} \frac{\varphi(\omega_0 + t_n d) - \varphi(\omega_0)}{t} \geqslant D_\omega L(x_0, \lambda, \omega_0) d.$$

As $\lambda \in \Lambda(x_0, \omega_0)$ can be chose arbitrarily, we conclude that

$$\liminf_{t \downarrow 0} \frac{\varphi(\omega_0 + t_n d) - \varphi(\omega_0)}{t} \geqslant \sup_{\lambda \in \Lambda(x_0, \omega_0)} D_\omega L(x_0, \lambda, \omega_0) d.$$

Together with (8) this implies that

$$\lim_{n \to \infty} \frac{\varphi(\omega_0 + t_n d) - \varphi(\omega_0)}{t_n} = \sup_{\lambda \in \Lambda(x_0, \omega_0)} D_\omega L(x_0, \lambda, \omega_0) d.$$

We obtain then that formula (6) holds. which completes the proof. □

Remark 3.1. Note that if T_0 is a positive semidefinite continuous linear self-adjoint operator on $\mathcal{H}$, then (QP_{ω_0}) is a convex problem and assumption (A) automatically satisfied. Consequently, the (A) can be dropped from the assumption of Theorem 3.1. if (QP_{ω_0}) is a convex problem.

4. Conclusions

By using the Legendre property of quadratic form, we established differentiability properties of the optimal value function for quadratic programming problems under linear constraints in Hilbert spaces.

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