ABELIAN CATEGORY OF COARTINIAN MODULES

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Abstract:
In this paper, we show that the category of $I$-coartinian modules forms an Abelian subcategory of the category of all $R$-modules provided that $\text{ara}(I) = 1$.

Keywords:
Coartinian module, cosupport, Koszul homology.
PHẠM TRỪ ABEL CỦA CÁC MÔDUN COARTIN

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Tóm tắt:  
Trong bài báo này, chúng tôi sẽ đưa ra một số điều kiện để lặp các môdun $I$-coartin tạo thành một phạm trừ con Abel của phạm trừ các $R$-môdun.

Từ khóa:  
Môdun coartin, đối giá, đồng điều Koszul.
1 INTRODUCTION

In this paper, \( R \) is a Noetherian commutative ring with identity, \( I \) is an ideal of \( R \) and \( M \) is an \( R \)-module. In [2], Hartshorne defined a module \( M \) to be I-cotinfeld if \( \text{Supp}_R M \subseteq V(I) \) and \( \text{Ext}^i_R(R/I, M) \) is finitely generated for all \( i \geq 0 \). He asked:

**Question.** Does the category \( \mathcal{M}(R, I)_{\text{cot}} \) of I-cotinform modules form an Abelian subcategory of the category of all \( R \)-modules? That is, if \( f : M \rightarrow N \) is an \( R \)-module homomorphism of I-cotinform modules, are \( \text{Ker } f \) and \( \text{Coker } f \) I-cotinform?

In [4], Nam introduced the \( I \)-coartinian modules which is in some sense dual to the concept of I-cotinform modules. An \( R \)-module \( M \) is said to be \( I \)-coartinian if \( \text{Cospur}_RM \subseteq V(I) \) and \( \text{Tor}^i_R(R/I, M) \) is an artinian \( R \)-module for all \( i \geq 0 \).

We recall that an \( R \) module \( L \) is called cocyclic if it is a submodule of the injective hull \( E(R/m) \) for some maximal ideal \( m \) of \( R \). In [8], Yassemi defined the cosupport of an \( R \)-module \( M \), denoted by \( \text{Cosupp}_R M \) to be the set of prime ideals \( p \) such that there exists a cocyclic homomorphic image \( L \) of \( M \) with \( \text{Ann}_R L \subseteq p \). If \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is a short exact sequence of \( R \)-modules then

\[
\text{Cosupp}_RB = \text{Cosupp}_RA \cup \text{Cosupp}_RC.
\]

**Question.** Does the category \( \mathcal{M}(R, I)_{\text{coa}} \) of \( I \)-coaartinian modules form an Abelian subcategory of the category of all \( R \)-modules? That is, if \( f : M \rightarrow N \) is an \( R \)-module homomorphism of \( I \)-coaartinian modules, are \( \text{Ker } f \) and \( \text{Coker } f \) \( I \)-coaartinian?

The main purpose of this paper is to provide a condition such that the category of \( I \)-coaartinian modules is Abelian. More precisely, we shall show that:

**Theorem.** Let \( I \) be an ideal of \( R \) such that \( \text{ara}(I) = 1 \). Then the category of \( I \)-coaartinian modules forms an Abelian subcategory of the category of all \( R \)-modules \( M \) satisfy \( IM = 0 \).

Throughout this paper, \( R \) will always be a commutative Noetherian ring with non-zero identity and \( I \) will be an ideal of \( R \). The radical of \( I \), denoted by \( \sqrt{I} \), is defined to be the set \( \{ x \in R \mid x^n \in I \text{ for some } n \gg 0 \} \).

2 MAIN RESULTS

First, we recall the definition of \( I \)-coaartinian modules.

**Definition 2.1.** ([4]) An \( R \)-module \( M \) is called \( I \)-coaartinian if \( \text{Cospur}_RM \subseteq V(I) \) and \( \text{Tor}^i_R(R/I, M) \) is artinian for all \( i \geq 0 \).

We also need some primary properties of \( I \)-coaartinian modules.

**Lemma 2.2.** ([4, Proposition 4.2]) The following statements hold:

(i) If \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is a short exact sequence and two of the modules are \( I \)-coaartinian, then so is the third one.

(ii) Let \( f : M \rightarrow N \) be a homomorphism of \( I \)-coaartinian modules. If one of the three modules \( \text{Ker } f, \text{Im } f \) and \( \text{Coker } f \) is \( I \)-coaartinian, then all three of them are \( I \)-coaartinian.

**Lemma 2.3.** ([4, Proposition 4.5]) Let \( I \) be an ideal of \( R \) and \( M \) is an \( I \)-coaartinian \( R \)-module. Then:

(i) \( \text{Tor}^i_R(N, M) \) is artinian for all \( i \geq 0 \) and any finitely generated \( R \)-module \( N \) such that \( I \subseteq \text{Ann}_R N \);

(ii) \( M \) is \( I^n \)-coaartinian for all integer \( n \geq 1 \);

(iii) For any ideal \( J \) of \( R \) such that \( \sqrt{J} = \sqrt{I} \), then \( M \) is \( J \)-coaartinian.

**Lemma 2.4.** Let \( I = (x_1, \ldots, x_n) \) be an ideal of \( R \) and \( M \) an \( R \)-module such that \( IM = 0 \). The following statements are equivalent:

(i) \( \text{Tor}^i_R(R/I, M) \) is artinian for all \( i \geq 0 \);

(ii) \( \text{Tor}^i_R(R/I, M) \) is artinian for all \( i = 0, 1, \ldots, n \);

(iii) The Koszul homology module \( H_i(x_1, \ldots, x_n; M) \) is artinian for all \( i = 0, 1, \ldots, n \).

**Proof.** (i) \( \Rightarrow \) (ii). Trivial.

(ii) \( \Rightarrow \) (iii). Consider the Koszul complex of \( M \) with respect to \( \underline{x} := x_1, \ldots, x_n \):

\[
K\star(\underline{x}; M) : 0 \rightarrow M_n \xrightarrow{\partial_n} M_{n-1} \rightarrow \cdots \rightarrow
M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{i_0} 0,
\]

where \( M_i = \bigoplus_{x \in \underline{x}} M_i(\underline{x}; M). \) It is clear that

\[
H_0(\underline{x}; M) = M/IM \cong R/I \otimes M
\]

and then \( H_0(\underline{x}; M) \) is artinian by the hypothesis. The short exact sequence

\[
0 \rightarrow \text{Im } \partial_1 \rightarrow \text{Ker } \partial_0 \rightarrow H_0(\underline{x}; M) \rightarrow 0
\]
induces a long exact sequence
\[
\text{Tor}^R_i(R/I, \text{Im} \partial_1) \rightarrow \text{Tor}^R_i(R/I, \text{Ker} \partial_0) \rightarrow \text{Tor}^R_i(R/I, H_0(x; M)) \rightarrow \cdots
\]
It should be mentioned that \( \text{Im} \partial_1 = IM \), therefore one gets \( \text{Tor}^R_i(R/I, \text{Im} \partial_1) = 0 \) for all \( i \geq 0 \). Moreover, applying the functor \( R/I \otimes_R - \) to the short exact sequence
\[
0 \rightarrow \text{Ker} \partial_1 \rightarrow M_1 \rightarrow \text{Im} \partial_1 \rightarrow 0
\]
we obtain isomorphisms
\[
\text{Tor}^R_i(R/I, \text{Ker} \partial_1) \cong \text{Tor}^R_i(R/I, M_1) \cong \oplus^n \text{Tor}_i(R/I, M)
\]
for all \( i \geq 0 \). By the assumption, \( \text{Tor}^R_i(R/I, \text{Ker} \partial_1) \) is artinian for all \( i = 0, 1, \ldots, n \). Next, the short exact sequence
\[
0 \rightarrow \text{Im} \partial_2 \rightarrow \text{Ker} \partial_1 \rightarrow H_1(x; M) \rightarrow 0
\]
induces that \( R/I \otimes_R H_1(x; M) \) is artinian. Since \( IH_1(x; M) = 0 \), it follows that \( H_1(x; M) \) is artinian. By the same method, we will prove that \( H_i(x; M) \) is artinian for all \( i = 2, \ldots, n \).

(iii) \( \Rightarrow \) (i). Let
\[
F_* : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0
\]
be a free resolution of finitely generated \( R \)-modules of \( R/I \). Next, consider the complex
\[
F_* \otimes_R M : \cdots \rightarrow F_{k+1} \otimes_R M \xrightarrow{d_{k+1}} F_k \otimes_R M \xrightarrow{d_k} \cdots
\]
and we have
\[
\text{Tor}^R_i(R/I, M) = H_i(F_* \otimes_R M)
\]
for each \( i \geq 0 \). We use induction to prove that \( H_i(x; \text{Ker} d_i) \) is artinian for all \( i \geq 0 \). Let \( i = 0 \), by the hypothesis, \( H_i(x; F_0 \otimes_R M) \) is artinian for all \( i \geq 0 \) since \( F_0 \otimes_R M \) is isomorphic to a finite copies of \( M \). Now, assume that \( k \geq 0 \) and \( H_i(x; \text{Ker} d_k) \) is artinian for all \( i \geq 0 \). The short exact sequence
\[
0 \rightarrow \text{Im} d_{k+1} \rightarrow \text{Ker} d_k \rightarrow \text{Tor}^R_i(R/I, M) \rightarrow 0
\]
induces the following exact sequence
\[
\text{Ker} d_k/I \text{Ker} d_k \rightarrow \text{Tor}^R_i(R/I, M) \rightarrow 0.
\]
Since \( H_0(x; \text{Ker} d_k) \cong \text{Ker} d_k/I \text{Ker} d_k \), we can conclude that \( \text{Tor}^R_i(R/I, M) \) is artinian. Moreover, this implies that \( H_i(x; \text{Im} d_{k+1}) \) is artinian for all \( i \geq 0 \). The short exact sequence
\[
0 \rightarrow \text{Ker} d_{k+1} \rightarrow F_{k+1} \otimes_R M \rightarrow \text{Im} d_{k+1} \rightarrow 0
\]
induces that \( H_i(x; \text{Ker} d_{k+1}) \) is artinian for all \( i \geq 0 \). By the similar arguments, we assert that \( \text{Tor}_{k+1}(R/I, M) \) is artinian and which completes the proof.

Let \( I \) be an ideal of \( R \). We recall that the arithmetic rank of \( I \), denoted by \( \text{ara}(I) \), is the least number of elements of \( I \) required to generate an ideal which has the same radical as \( I \), i.e.,
\[
\text{ara}(I) = \min \{ n \mid \text{there exists } x_1, \ldots, x_n \in I \text{ such that } \sqrt{(x_1, \ldots, x_n)} = \sqrt{I} \}.
\]

**Theorem 2.5.** Let \( M \) be a non-zero \( R \)-module such that \( IM = 0 \). Then the following conditions are equivalent:

(i) \( \text{Tor}^R_i(R/I, M) \) is artinian for all \( i \geq 0 \);

(ii) \( \text{Tor}^R_i(R/I, M) \) is artinian for all \( i = 0, 1, \ldots, \text{ara}(I) \).

**Proof.** It follows from Lemma 2.4.

**Corollary 2.6.** Let \( M \) be a non-zero \( R \)-module with \( IM = 0 \) and \( \text{Cosupp}_R M \subseteq V(I) \). Then the following conditions are equivalent:

(i) \( M \) is \( I \)-coartinian;

(ii) \( \text{Tor}^R_i(R/I, M) \) is artinian for all \( i = 0, 1, \ldots, \text{ara}(I) \).

Now, we are going to state and prove the main result of this paper.

**Theorem 2.7.** Let \( I \) be an ideal of \( R \) such that \( \text{ara}(I) = 1 \). Then the category of \( I \)-coartinian modules \( M \) with \( IM = 0 \) forms an Abelian subcategory of the category of all \( R \)-modules.

**Proof.** Let \( M, N \) be two \( I \)-coartinian \( R \)-modules such that \( IM = IN = 0 \) and \( f : M \rightarrow N \) an \( R \)-homomorphism. It is enough to show that the \( R \)-modules \( \text{Ker} f \) and \( \text{Coker} f \) are \( I \)-coartinian. The short exact sequences
\[
0 \rightarrow \text{Ker} f \rightarrow M \rightarrow \text{Im} f \rightarrow 0
\]
and
\[
0 \rightarrow \text{Im} f \rightarrow N \rightarrow \text{Coker} f \rightarrow 0
\]
induce the following exact sequences
\[
\cdots \rightarrow \text{Tor}^R_2(R/I, \text{Im} f) \rightarrow \text{Tor}^R_1(R/I, \text{Ker} f) \rightarrow \text{Tor}^R_1(R/I, M) \rightarrow \cdots
\]
}\[
\]
Tor_1^R(R/I, \text{Im } f) \to \text{Ker } f / I \text{Ker } f \to M / IM \to \\
\text{Im } f / I \text{Im } f \to 0

and

\cdots \to \text{Tor}_2^R(R/I, \text{Coker } f) \to \text{Tor}_1^R(R/I, \text{Im } f) \to \\
\to \text{Tor}_1^R(R/I, N) \to \cdots

\text{Tor}_1^R(R/I, \text{Coker } f) \to \text{Im } f / I \text{Im } f \to N / IN \\
\to \text{Coker } f / I \text{Coker } f \to 0.

Since \( M, N \) are both \( I \)-coartinian \( R \)-modules, it follows that \( \text{Ker } f / I \text{Ker } f \), \( \text{Coker } f / I \text{Coker } f \), \( \text{Tor}_1^R(R/I, \text{Ker } f) \) and \( \text{Tor}_1^R(R/I, \text{Coker } f) \) are artinian. Hence, the conclusion follows from Corollary 2.6.

3 CONCLUSION

In this paper, we showed some conditions to module \( \text{Tor}_1^R(R/I, M) \) is artinian. In particular, we gave a condition such that the category of \( I \)-coartinian modules is Abelian.

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