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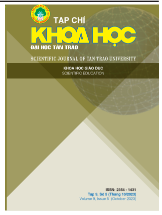

ABELIAN CATEGORY OF COARTINIAN MODULES

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Abstract:

In this paper, we show that the category of I-coartinian modules forms an Abelian subcategory of the category of all R-modules provided that $\operatorname{ara}(I) = 1$.

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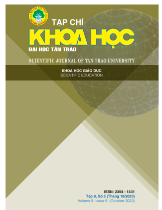

PHẠM TRÙ ABEL CỦA CÁC MÔĐUN COARTIN

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Từ khóa:
Môđun coartin, đối giá, đồng điều Koszul.

Tóm tắt:

Trong bài báo này, chúng tôi sẽ đưa ra một số điều kiện để lớp các môđun I-coartin tạo thành một phạm trù con Abel của phạm trù các $R-$môđun.

1 INTRODUCTION

In this paper, R is a Noetherian commutative ring with identity, I is an ideal of R and M is an R-module. In [2], Hartshorne defined a module M to be I-cofinite if $\mathrm{Supp}_R M \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. He asked:

Question. Does the category $\mathcal{M}(R,I)_{cof}$ of I-cofinite modules form an Abelian subcategory of the category of all R-modules? That is, if $f : M \to N$ is an R-module homomorphism of I-cofinite modules, are $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ I-cofinite?

In [4], Nam introduced the I-coartinian modules which is in some sense dual to the concept of I-cofinite modules. An R-module M is said to be I-coartinian if $\mathrm{Cosupp}_R M \subseteq V(I)$ and $\mathrm{Tor}^R_i(R/I, M)$ is an artinian R-module for all $i \geq 0$.

We recall that an R module L is called *cocyclic* if it is a submodule of the injective hull $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. In [8], Yassemi defined the *cosupport* of an R-module M, denoted by $\mathrm{Cosupp}_R M$ to be the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}_R L \subseteq \mathfrak{p}$. If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules then

$$\mathrm{Cosupp}_R B = \mathrm{Cosupp}_R A \cup \mathrm{Cosupp}_R C.$$

Question. Does the category $\mathcal{M}(R,I)_{coa}$ of I-coartinian modules form an Abelian subcategory of the category of all R-modules? That is, if $f : M \to N$ is an R-module homomorphism of I-coartinian modules, are $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ I-coartinian?

The main purpose of this paper is to provide a condition such that the category of I-coartinian modules is Abelian. More precisely, we shall show that:

Theorem. Let I be an ideal of R such that $\mathrm{ara}(I) = 1$. Then the category of I-coartinian modules forms an Abelian subcategory of the category of all R-modules M satisfy $IM = 0$.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. The radical of I, denoted by $\sqrt{I}$, is defined to be the set $\{x \in R \mid x^n \in I \text{ for some } n \gg 0\}$.

2 MAIN RESULTS

First, we recall the definition of I-coartinian modules.

Definition 2.1. ([4]) An R-module M is called I-coartinian if $\mathrm{Cosupp}_R(M) \subseteq V(I)$ and $\mathrm{Tor}^R_i(R/I, M)$ is artinian for all $i \geq 0$.

We also need some primary properties of I-coartinian modules.

Lemma 2.2. ([4, Proposition 4.2]) The following statements hold:

(i) If $0 \to A \to B \to C \to 0$ is a short exact sequence and two of the modules are I-coartinian, then so is the third one.

(ii) Let $f : M \to N$ be a homomorphism of I-coartinian modules. If one of the three modules $\mathrm{Ker}\, f, \mathrm{Im}\, f$ and $\mathrm{Coker}\, f$ is I-coartinian, then all three of them are I-coartinian.

Lemma 2.3. ([4, Proposition 4.5]) Let I be an ideal of R and M is an I-coartinian R-module. Then:

(i) $\mathrm{Tor}^R_i(N, M)$ is artinian for all $i \geq 0$ and any finitely generated R-module N such that $I \subseteq \mathrm{Ann}_R N$;

(ii) M is I^n-coartinian for all integer $n \geq 1$;

(iii) For any ideal J of R such that $\sqrt{J} = \sqrt{I}$, then M is J-coartinian.

Lemma 2.4. Let $I = (x_1, \ldots, x_n)$ be an ideal of R and M an R-module such that $IM = 0$. The following statements are equivalent:

(i) $\mathrm{Tor}^R_i(R/I, M)$ is artinian for all $i \geq 0$;

(ii) $\mathrm{Tor}^R_i(R/I, M)$ is artinian for all $i = 0, 1, \ldots, n$;

(iii) The Koszul homology module $H_i(x_1, \ldots, x_n; M)$ is artinian for all $i = 0, 1, \ldots, n$.

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Consider the Koszul complex of M with respect to $\underline{x} := x_1, \ldots, x_n$

$$K_\bullet(\underline{x}; M) : 0 \to M_n \overset{\partial_n}{\to} M_{n-1} \to \cdots \to$$
$$\to M_1 \overset{\partial_1}{\to} M_0 \overset{\partial_0}{\to} 0,$$

where $M_i = \bigoplus^{C^i_n} M$. It is clear that

$$H_0(\underline{x}; M) = M/IM \cong R/I \otimes M$$

and then $H_0(\underline{x}; M)$ is artinian by the hypothesis. The short exact sequence

$$0 \to \mathrm{Im}\, \partial_1 \to \mathrm{Ker}\, \partial_0 \to H_0(\underline{x}; M) \to 0$$

induces a long exact sequence

$$\operatorname{Tor}_i^R(R/I, \operatorname{Im}\partial_1) \to \operatorname{Tor}_i^R(R/I, \operatorname{Ker}\partial_0) \to \to \operatorname{Tor}_i^R(R/I, H_0(\underline{x}; M)) \to \cdots .$$

It should be mentioned that $\operatorname{Im}\partial_1 = IM$, therefore one gets $\operatorname{Tor}_i^R(R/I, \operatorname{Im}\partial_1) = 0$ for all $i \geq 0$. Moreover, applying the functor $R/I \otimes_R -$ to the short exact sequence

$$0 \to \operatorname{Ker}\partial_1 \to M_1 \to \operatorname{Im}\partial_1 \to 0$$

we obtain isomorphisms

$$\operatorname{Tor}_i^R(R/I, \operatorname{Ker}\partial_1) \cong \operatorname{Tor}_i^R(R/I, M_1) \cong \oplus^n \operatorname{Tor}_i(R/I, M)$$

for all $i \geq 0$. By the assumption, $\operatorname{Tor}_i^R(R/I, \operatorname{Ker}\partial_1)$ is artinian for all $i = 0, 1, \ldots, n$. Next, the short exact sequence

$$0 \to \operatorname{Im}\partial_2 \to \operatorname{Ker}\partial_1 \to H_1(\underline{x}; M) \to 0$$

induces that $R/I \otimes_R H_1(\underline{x}; M)$ is artinian. Since $IH_1(\underline{x}; M) = 0$, it follows that $H_1(\underline{x}; M)$ is artinian. By the same method, we will prove that $H_i(\underline{x}; M)$ is artinian for all $i = 2, \ldots, n$.

(iii) $\Rightarrow$ (i). Let

$$F_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to 0$$

be a free resolution of finitely generated R-modules of R/I. Next, consider the complex

$$F_\bullet \otimes_R M : \cdots \to F_{k+1} \otimes_R M \stackrel{d_{k+1}}{\to} F_k \otimes_R M \stackrel{d_k}{\to} \cdots$$

and we have

$$\operatorname{Tor}_i^R(R/I, M) = H_i(F_\bullet \otimes_R M)$$

for each $i \geq 0$. We use induction to prove that $H_i(\underline{x}; \operatorname{Ker} d_i)$ is artinian for all $i \geq 0$. Let $i = 0$, by the hypothesis, $H_i(\underline{x}; F_0 \otimes_R M)$ is artinian for all $i \geq 0$ since $F_0 \otimes_R M$ is isomorphic to a finite copies of M. Now, assume that $k \geq 0$ and $H_i(\underline{x}; \operatorname{Ker} d_k)$ is artinian for all $i \geq 0$. The short exact sequence

$$0 \to \operatorname{Im} d_{k+1} \to \operatorname{Ker} d_k \to \operatorname{Tor}_k^R(R/I, M) \to 0$$

induces the following exact sequence

$$\operatorname{Ker} d_k / I \operatorname{Ker} d_k \to \operatorname{Tor}_k^R(R/I, M) \to 0.$$

Since $H_0(\underline{x}; \operatorname{Ker} d_k) \cong \operatorname{Ker} d_k / I \operatorname{Ker} d_k$, we can conclude that $\operatorname{Tor}_k^R(R/I, M)$ is artinian. Moreover, this implies that $H_i(\underline{x}; \operatorname{Im} d_{k+1})$ is artinian for all $i \geq 0$. The short exact sequence

$$0 \to \operatorname{Ker} d_{k+1} \to F_{k+1} \otimes_R M \to \operatorname{Im} d_{k+1} \to 0$$

induces that $H_i(\underline{x}; \operatorname{Ker} d_{k+1})$ is artinian for all $i \geq 0$. By the similar arguments, we assert that $\operatorname{Tor}_{k+1}^R(R/I, M)$ is artinian and which completes the proof.

Let I be an ideal of R. We recall that the arithmetic rank of I, denoted by $\operatorname{ara}(I)$, is the least number of elements of I required to generate an ideal which has the same radical as I, i.e.,

$$\operatorname{ara}(I) = \min\{n \mid \text{there exists } x_1, \ldots, x_n \in I \text{ such that } \sqrt{(x_1, \ldots, x_n)} = \sqrt{I}\}.$$

Theorem 2.5. Let M be a non-zero R-module such that $IM = 0$. Then the following conditions are equivalent:

(i) $\operatorname{Tor}_i^R(R/I, M)$ is artinian for all $i \geq 0$;

(ii) $\operatorname{Tor}_i^R(R/I, M)$ is artinian for all $i = 0, 1, \ldots, \operatorname{ara}(I)$.

Proof. It follows from Lemma 2.4.

Corollary 2.6. Let M be a non-zero R-module with $IM = 0$ and $\operatorname{Cosupp}_R M \subseteq V(I)$. Then the following conditions are equivalent:

(i) M is I-coartinian;

(ii) $\operatorname{Tor}_i^R(R/I, M)$ is artinian for all $i = 0, 1, \ldots, \operatorname{ara}(I)$.

Now, we are going to state and prove the main result of this paper.

Theorem 2.7. Let I be an ideal of R such that $\operatorname{ara}(I) = 1$. Then the category of I-coartinian modules M with $IM = 0$ forms an Abelian subcategory of the category of all R-modules.

Proof. Let M, N be two I-coartinian R-modules such that $IM = IN = 0$ and $f : M \to N$ an R-homomorphism. It is enough to show that the R-modules $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are I-coartinian. The short exact sequences

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$$

and

$$0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0$$

induce the following exact sequences

$$\cdots \to \operatorname{Tor}_2^R(R/I, \operatorname{Im} f) \to \operatorname{Tor}_1^R(R/I, \operatorname{Ker} f) \to \to \operatorname{Tor}_1^R(R/I, M) \to \cdots$$

$$\operatorname{Tor}_1^R(R/I, \operatorname{Im} f) \to \operatorname{Ker} f/I \operatorname{Ker} f \to M/IM \to$$
$$\to \operatorname{Im} f/I \operatorname{Im} f \to 0$$

and

$$\cdots \to \operatorname{Tor}_2^R(R/I, \operatorname{Coker} f) \to \operatorname{Tor}_1^R(R/I, \operatorname{Im} f) \to$$
$$\to \operatorname{Tor}_1^R(R/I, N) \to \cdots$$

$$\operatorname{Tor}_1^R(R/I, \operatorname{Coker} f) \to \operatorname{Im} f/I \operatorname{Im} f \to N/IN \to$$
$$\to \operatorname{Coker} f/I \operatorname{Coker} f \to 0.$$

Since M, N are both I-coartinian R-modules, it follows that $\operatorname{Ker} f/I \operatorname{Ker} f$, $\operatorname{Coker} f/I \operatorname{Coker} f, \operatorname{Tor}_1^R(R/I, \operatorname{Ker} f)$ and $\operatorname{Tor}_1^R(R/I, \operatorname{Coker} f)$ are artinian. Hence, the conclusion follows from Corollary 2.6.

3 CONCLUSION

In this paper, we showed some conditions to module $\operatorname{Tor}_i^R(R/I, M)$ is artinian. In particular, we gave a condition such that the category of I-coartinian modules is Abelian.

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