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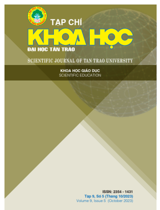

A PROJECTION ALGORITHM FOR FINDING A COMMON SOLUTION OF EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract:

In this paper, we design a new projection algorithm for finding a common solution of equilibrium and fixed point problems in a real Hilbert space. The proposed algorithm is a combination of the projection method and Man iterative technique. Furthermore, the algorithm uses self-adaptive sizes at each iteration. The convergent theorem is established under mild conditions. We also apply the proposed algorithms to solve a oligopolistic Nash-Cournot equilibrium model.

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THUẬT TOÁN CHIẾU DƯỚI VI PHÂN TÌM NGHIỆM CHUNG CỦA BÀI TOÁN CÂN BẰNG VÀ BÀI TOÁN ĐIỂM BẤT ĐỘNG

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Thông tin bài viết

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Từ khóa:
Bài toán cân bằng, liên tục Lipschitz, giả đơn điệu, phương pháp chiếu, phương pháp dưới vi phân, bài toán điểm bất động.

Tóm tắt:

Trong bài báo này, chúng tôi thiết kế một thuật toán chiếu dưới vi phân mới để tìm nghiệm chung của bài toán cân bằng và bài toán điểm bất động trong không gian Hilbert thực. Thuật toán được đề xuất là sự kết hợp giữa phương pháp chiếu, phương pháp dưới đạo hàm và kỹ thuật lặp Man. Hơn nữa, thuật toán của chúng tôi sử dụng các bước lặp tự thích ứng ở mỗi lần lặp. Chúng tôi chứng minh được thuật toán hội tụ với các giả thiết nhẹ. Chúng tôi cũng áp dụng thuật toán đề xuất để giải mô hình cân bằng Nash-Cournot.

1 INTRODUCTION

Let $\mathbb{H}$ denote a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. The equilibrium problem, shortly (EPs), for the bifunction f on a nonempty convex set C is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0 \quad \forall y \in C,$$

where $f : C \times C \to \mathbb{R}$ is a bifunction such that $f(x,x) = 0$ for all $x \in C$. In the framework of this paper, we denote the solution set of Problem (EPs) by $Sol(EPs)$. Problem (EPs) is a general model of some important mathematical models such as optimization, variational inequality, Kakutani fixed point, and so on (see, for example, [1, 4]). Therefore, the problem has received a lot of research attention from mathematicians. In order to solve (EPs), many iterative methods have been proposed, among them, the projection and the extragradient (or double projection) algorithms are widely used (see [2, 5, 10] and the references therein). In [5], authors introduced a projection algorithm, that only uses one projection, for an equilibrium problem involving pseudomonotone continuous bifunction f such that its diagonal subdifferential is Lipschitz continuous. The strongly convergent theorems are established under standard assumptions.

Motivated and inspired by the projection method in [5] and the Man iteration technique for fixed point problems, we design a new projection algorithm for finding a common element of the solution sets of Problem (EPs) and the set of fixed points of a demicontractive mappings S, namely:

$$\text{Find } x^* \in \Omega = Fix(S) \cap Sol(EPs).$$

Furthermore, the algorithm uses self-adaptive sizes at each iteration. We have proved that the proposed algorithm is strongly convergent under the mild assumptions. We also apply the proposed algorithms to solve a modified oligopolistic Nash-Cournot equilibrium model.

The remaining part of the paper is organized as follows. Section 2 shows preliminaries, some lemmas that will be used in proving the convergence of our proposed algorithm. The proposed algorithm and its convergence analysis are presented in Section 3.

2 PRELIMINARIES

In this section, we recall some concepts and results that are used to prove the main results of this paper. For any subsets A and B of $\mathbb{H}$, the *Hausdorff distance* between these subsets is defined by

$$\rho(A, B) := \max\{d(A, B), d(B, A)\}$$

where $d(A, B) := \sup_{a\in A} \inf_{b\in B} \|a - b\|$.

The metric projection from $\mathbb{H}$ onto C is denoted by P_C and

$$P_C(x) = argmin\{\|x - y\| : \ y \in C\} \ x \in \mathbb{H}.$$

From the definition of projection, it is easy to see that P_C has the following characteristic properties.

Lemma 2.1. *For any $x \in \mathbb{H}$, we have*

(i) *$p = P_C(x)$ if and only if $\langle p - x, y - p\rangle \leq 0, \quad \forall y \in C$;*

(ii) $\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{H}$.

Definition 2.2. A bifunction $f : C \times C \to \mathbb{H}$ is called to be

(*i*) *monotone on* C, if $f(x, y) + f(y, x) \leq 0 \ \forall x, y \in C$;

(*ii*) *pseudomonotone on* C, if $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \ \forall x, y \in C$.

Definition 2.3. Let $C \subset \mathbb{H}$ be a nonempty subset. An operator $S : C \to \mathbb{H}$ is called to be

(i) *β-demi-contractive* on C, if $Fix(S)$ is nonempty and there exists $\beta \in [0, 1)$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + \beta\|x - Sx\|^2, \qquad (1)$$

for all $x \in C$ and $p \in Fix(S)$;

(ii) *demi-closed*, if for any sequence $\{x^k\} \subset C$, $x^k \rightharpoonup z \in C$, $(I - S)(x^k) \rightharpoonup 0$ implies $z \in Fix(S)$.

It is well known that if S is β-demi-contractive on C then S is demi-closed and (1) is equivalent to (see [8])

$$\langle x - Sx, x - p\rangle \geq \frac{1}{2}(1 - \beta)\|x - Sx\|^2, \qquad (2)$$

for all $x \in C$ and $p \in Fix(S)$.

The following lemmas are useful in the sequel.

Lemma 2.4. *([7]) Let $\{\xi_k\}$ be a sequence of nonnegative real numbers satisfying the following condition*

$$\xi_{k+1} \leq (1 - \varrho_k)\xi_k + \varrho_k\alpha_k + \beta_k, \ \forall k \geq 1,$$

where $\{\varrho_k\} \subset [0, 1]$, $\sum_{k=0}^{\infty} \varrho_k = +\infty$, $\limsup_{k\to\infty} \alpha_k \leq 0$ and $\beta_k \geq 0$, $\sum_{n=1}^{\infty} \beta_k < \infty$. Then, $\lim\limits_{k\to\infty} \xi_k = 0$.

Lemma 2.5. *([7], Remark 4.4) Let $\{\xi_k\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer M such that $M \geq m$ and $\xi_M \leq \xi_{M+1}$. Let $\bar{k}$ be an integer such that $\xi_{\bar{k}} \leq \xi_{\bar{k}+1}$ and define, for all integer $k \geq \bar{k}$,*

$$\tau(k) = \max\{i \in \mathbb{N} : \bar{k} \leq i \leq k, \xi_i \leq \xi_{i+1}\}.$$

Then, $0 \leq \xi_k \leq \xi_{\tau(k)+1}$ for all $k \geq \bar{k}$ the and sequence $\{\tau(k)\}_{k\geq\bar{k}}$ is nondecreasing and tends to $+\infty$ as $k \to \infty$.

3 PROJECTION ALGORITHM

Let us assume that the bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ and mappings S satisfy the following conditions:

A_1. $f(x, x) = 0$ for all $x \in C$, $f(x, y)$ is pseudomonotone on $C \times C$ and $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C;

A_2. $S : \mathbb{H} \to \mathbb{H}$ is β-demicontractive and demiclosed;

A_3. the set Ω is nonempty;

A_4. there exists a real positive number L such that

$$\rho\Big(\partial_2 f(x, \cdot)(x), \partial_2 f(y, \cdot)(y)\Big) \leq L\|x - y\|,$$

for all $x, y \in C$, where $\partial_2 f(x, \cdot)(x)$ is subdifferential of $f(x, \cdot)$ at x, i.e.,

$$\partial_2 f(x, \cdot)(x) = \{\xi \in \mathbb{H} : \ \langle \xi, z-y\rangle \leq f(x, z), \ \forall z \in C\}.$$

Now, we describe our approximate projection algorithm.

Algorithm 3.1. *Take arbitrary starting point* $x^0 \in C$, $\gamma_0 > 0$, $0 < \nu < 1$, $\overline{L} > L$ *and control parameter sequences* $\{\varrho_k\}$, $\{\nu_k\}$, $\{\gamma_k\}$, $\{\mu_k\}$ *satisfying conditions:*

$$\begin{cases} 0 < \mu_k, \ \sum_{k=0}^{+\infty} \mu_k < +\infty \\ \varrho_k \in (0,1), \ \lim_{k\to\infty} \varrho_k = 0, \ \sum_{k=0}^{+\infty} \varrho_k = +\infty \\ \nu_k \in [0,1), \ \lim_{k\to\infty} \frac{\nu_k}{\varrho_k} = 0, \sum_{k=0}^{+\infty} \nu_k < +\infty. \end{cases} \tag{3}$$

Step 1. $(k = 0, 1, ...)$ *Choose* $u^k \in \partial_2 f(x^k, x^k)$ *and compute* $y^k = P_C(x^k - \gamma_k u^k)$.

Step 2. *Take* $v^k \in B\left(u^k, \overline{L}\|x^k - y^k\|\right) \cap \partial_2 f(y^k, y^k)$, *where* $B\left(u^k, \overline{L}\|x^k - y^k\|\right) := \{u \in \mathbb{H} : \|u - u^k\| \leq \overline{L}\|x^k - y^k\|\}$. *Compute* $z^k = (1 + \theta_k)y^k - \theta_k x^k + \gamma_k(u^k - v^k)$, *where*

$$\theta_k = \begin{cases} \min\{\frac{\nu_k}{\|u^k\|\|x^k - y^k\|}, \nu_k\}, & \text{if } x^k - y^k \neq 0, \\ \nu_k & \text{otherwise.} \end{cases}$$

Compute

$$\gamma_{k+1} = \begin{cases} \min\left\{\frac{\nu\|x^k - y^k\|}{\|u^k - v^k\|}, \gamma_k + \mu_k\right\}, & \text{if } u^k - v^k \neq 0, \\ \gamma_k + \mu_k & \text{otherwise.} \end{cases}$$

Step 3. *Compute* $w^k = \varrho_k x^0 + (1 - \varrho_k)z^k$,

$$x^{k+1} = (1 - \omega)w^k + \omega S w^k, \ 0 < \omega < 1 - \beta.$$

Step 4. *Set* $k := k + 1$, *and go to Step 1.*

We first obtain the following important lemma.

Lemma 3.1. *Assume that* $(A_1) - (A_4)$ *hold. Let* $p \in Sol(EPs)$ *and* $\{x^k\}$, $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, $\{\gamma_k\}$ *be the sequences generated by Algorithm 3.1. Then,*

(i) $\gamma_k \in [\min\{\frac{\nu}{L}, \gamma_0\}, \ \gamma_0 + M]$, $\forall k \geq 0$ *and* $\lim_{k\to\infty} \gamma_k = \lambda$, *where* $\sum_{k=0}^{+\infty} \mu_k = M$;

(ii) $\|z^k - p\|^2 \leq \|x^k - p\|^2 + 2(\gamma_0 + M)\nu_k - \left[1 - \left(\theta_k + \gamma_k \frac{\nu}{\gamma_{k+1}}\right)^2\right] \|y^k - x^k\|^2$;

(iii) *the sequences* $\{x^k\}$, $\{y^k\}$, $\{z^k\}$, $\{w^k\}$ *and* $\{u^k - v^k\}$ *are bounded.*

Proof. By reasoning similar to the proof of Lemma 3.4 in [5], we have (i) and (ii).

Now we prove (iii). By Step 3 and the β demicontractive assumption of S, we get

$$\begin{aligned} &||x^{k+1} - p||^2 = ||(1 - \omega)w^k + \omega S w^k - p||^2 \\ =&||(w^k - p) + \omega(Sw^k - w^k)||^2 \\ \leq&||w^k - p||^2 + 2\omega\langle w^k - p, Sw^k - w^k\rangle \\ &+ \omega^2||Sw^k - w^k||^2 \\ \leq&||w^k - p||^2 + \omega(\omega + \beta - 1)||Sw^k - w^k||^2 \\ \leq&||w^k - p||^2. \end{aligned} \tag{4}$$

We have from (i) and Condition (3) that

$$\lim_{k\to\infty} \left[1 - \left(\theta_k + \gamma_k \frac{\nu}{\gamma_{k+1}}\right)^2\right] = 1 - \nu^2 > 0,$$

which implies that there exist a nonnegative integer K_0 such that

$$1 - \left(\theta_k + \gamma_k \frac{\nu}{\gamma_{k+1}}\right)^2 > 0, \ \forall k \geq K_0.$$

From the above inequality and (ii), it follows that

$$\|z^k - p\|^2 \leq \|x^k - p\|^2 + 2(\gamma_0 + M)\nu_k, \ \forall k \geq K_0.$$

From the above inequality, the definition of w^k and (4), for every $k \geq K_0$, we have

$$\begin{aligned} &\|x^{k+1} - p\|^2 \leq \|w^k - p\|^2 \\ &= \|\varrho_k x^0 + (1 - \varrho_k)z^k - p\|^2 \\ &\leq \varrho_k\|x^0 - p\|^2 + (1 - \varrho_k)\|z^k - p\|^2 \\ &\leq \varrho_k\|x^0 - p\|^2 + (1 - \varrho_k)(\|x^k - p\|^2 + 2(\gamma_0 + M)\nu_k) \\ &\leq \max\left\{\|x^0 - p\|^2, \|x^k - p\|^2 + A_k\right\}, \end{aligned} \tag{5}$$

where $A_k = 2(\gamma_0 + M)\nu_k$ for every $k \geq K_0$. Similarly, we have

$$\|x^k - p\|^2 \leq \max\{\|x^0 - p\|^2, \|x^{k-1} - p\|^2 + A_{k-1}\}.$$

This together with (5) implies that

$$\begin{aligned} &\|x^{k+1} - p\|^2 \leq \\ &\max\left\{\|x^0 - p\|^2 + A_k, \|x^{k-1} - p\|^2 + A_{k-1} + A_k\right\} \\ &\cdots \\ &\leq \max\left\{\|x^0 - p\|^2, \ \|x^{K_0} - p\|^2\right\} + \sum_{k=K_0}^{\infty} A_k \\ &< +\infty, \end{aligned}$$

where the latest equality holds because $\sum_{k=K_0}^{\infty} A_k < +\infty$. This implies that $\{x^k\}$ is bounded. By (ii), for all $k \geq K_0$, we have

$$\|z^k - p\|^2 \leq \|x^k - p\|^2 + 2(\gamma_0 + M)\nu_k \tag{6}$$

and

$$\left[\left(\theta_k + \gamma_k \frac{\nu}{\gamma_{k+1}}\right)^2 - 1\right] \|y^k - x^k\|^2 \leq \|x^k - p\|^2 - \|z^k - p\|^2 + 2(\gamma_0 + M)\nu_k.$$

It follows from (6) and Condition (3) that $\{z^k\}$ is bounded. This together with the last inequality and the boundedness of $\{x^k\}$ implies that $\{y^k\}$ is bounded. We have from (5) that $\{w^k\}$ is bounded. Finally, we can deduce from $v^k \in B\left(u^k, \overline{L}\|x^k - y^k\|\right)$ that

$$\|u^k - v^k\| \leq \overline{L}\|x^k - y^k\|,$$

and so the sequence $\{u^k - v^k\}$ is bounded. □

Lemma 3.2. *Assume that $(A_1) - (A_4)$ hold. Let $\lim_{k\to\infty} \|x^k - y^k\| = 0$, $\lim_{k\to\infty} \|z^k - y^k\| = 0$, $\lim_{k\to\infty} \|x^{k+1} - w^k\| = 0$ and a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ converge weakly to p. Then, $p \in \Omega$.*

Proof. Since $\|x^k - y^k\| \to 0$ and the subsequence $\{x^{k_i}\}$ converges weakly to p, the sequence $\{y^{k_i}\}$ also converges weakly to p. From $v^k \in B\left(u^k, \overline{L}\|x^k - y^k\|\right)$, it follows that

$$\|u^k - v^k\| \leq \overline{L}\|x^k - y^k\|,$$

and so $\lim_{k\to\infty} \|u^k - v^k\| = 0$. We get from **Step 2** that

$$\langle y^{k_i} - x^{k_i} + \gamma_{k_i} u^{k_i}, x - y^{k_i}\rangle \geq 0 \ \forall x \in C,$$

which together with $u^{k_i} \in \partial_2 f(x^{k_i}, x^{k_i})$ implies that

$$\begin{aligned}
&\langle x^{k_i} - y^{k_i}, x - y^{k_i}\rangle \leq \gamma_{k_i}\langle u^{k_i}, x - y^{k_i}\rangle + \epsilon_{k_i}\\
\leq&\gamma_{k_i}(\langle v^{k_i}, x - y^{k_i}\rangle + \langle u^{k_i} - v^{k_i}, x - y^{k_i}\rangle) + \epsilon_{k_i}\\
\leq&\gamma_{k_i} f(y^{k_i}, x) + \gamma_{k_i}\langle u^{k_i} - v^{k_i}, x - y^{k_i}\rangle + \epsilon_{k_i}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{\gamma_{k_i}}\langle x^{k_i} - y^{k_i}, x - y^{k_i}\rangle &\leq f(y^{k_i}, x)\\
&+ \langle u^{k_i} - v^{k_i}, x - y^{k_i}\rangle + \frac{1}{\gamma_{k_i}}\epsilon_{k_i}.
\end{aligned}$$

For each fixed point $x \in C$, taking the limit as $i \to \infty$ on both sides of the last inequality, using $\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0$, $\lim_{i\to\infty} \|u^{k_i} - v^{k_i}\| = 0$, the weak upper semicontinuity of the function $f(\cdot, y)$ and the boundedness of the sequence $\{y^k\}$, we get

$$f(p, x) \geq 0 \quad \forall x \in C.$$

It means that $p \in Sol(EPs)$.

We now show that $p \in Fix(S)$. Using Step 3, we have

$$||w^k - Sw^k|| = \frac{1}{\omega}||x^{k+1} - w^k||.$$

From $\lim_{k\to\infty} \|x^{k+1} - w^k\| = 0$ and last equality, it follows that $||w^k - Sw^k|| \to 0, \ k \to \infty$. Also we know from Step 3 that

$$||w^k - z^k|| = \alpha_k||x^0 - z^k|| \leq \alpha_k M_0 \to 0, \ k \to \infty, \tag{7}$$

where $M_0 = \sup\{||x^0 - z^k|| : \ k = 0, 1, ...\}$. Using $\lim_{k\to\infty} \|x^k - y^k\| = 0$, $\lim_{k\to\infty} \|z^k - y^k\| = 0$ and $\|z^k - x^k\| \leq \|z^k - y^k\| + \|y^k - x^k\|$, we have $\lim_{k\to\infty} \|z^k - x^k\| = 0$. Combining this and (7), we obtain

$$\|w^k - x^k\| \leq \|w^k - z^k\| + \|z^k - x^k\| \to 0, \ k \to \infty.$$

From this and $x^{k_i} \rightharpoonup z$, it follows that $w^{k_i} \rightharpoonup p$. Using this, $\lim_{k\to\infty} ||w^k - Sw^k|| = 0$ and the demiclosedness of S, we have $p \in Fix(S)$. □

Now we state and prove the main convergence result of the algorithm in the following theorem.

Theorem 3.3. *Let bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfy the assumptions $(A_1) - (A_4)$. Then, the sequence $\{x^k\}$ generated by Algorithm 3.1 converges strongly to a solution $p \in \Omega$, where $p = P_\Omega(x^0)$.*

Proof. Set $\xi_k = \|x^k - p\|^2$, $\alpha_k = 2\langle x^0 - p, w^k - p\rangle$ and $\beta_k = 2(\gamma_0 + M)\nu_k$. To prove this theorem, we consider two following cases.

Case 1. Suppose that there exists $\bar{k} \in \mathbb{N}$ such that $\xi_{k+1} \leq \xi_k$ for all $k \geq \bar{k}$. Then, there exists the limit $\lim_{k\to\infty} \xi_k \in [0, \infty)$. Using Step 3, we obtain

$$\begin{aligned}
&\|x^{k+1} - p\|^2 = \|(1 - \omega)w^k + \omega Sw^k - p\|^2\\
=&\|w^k - p\|^2 - 2\omega\langle w^k - p, w^k - Sw^k\rangle\\
&+ \omega^2\|w^k - Sw^k\|^2.
\end{aligned} \tag{8}$$

which together with Lemma 3.1 and (2) implies that

$$\begin{aligned}
&\|x^{k+1}-p\|^2\\
\le&\|w^k-p\|^2-\omega(1-\beta-\omega)\|w^k-Sw^k\|^2\\
=&||\varrho_k(x^0-p)+(1-\varrho_k)(z^k-p)||^2\\
&-\frac{1}{\omega}(1-\beta-\omega)\|x^{k+1}-w^k\|^2\\
\le&(1-\varrho_k)||z^k-p||^2+2\varrho_k\langle x^0-p,w^k-p\rangle\\
&-\|x^{k+1}-w^k\|^2,\\
\le&||z^k-p||^2+2\varrho_k\langle x^0-p,w^k-p\rangle-\|x^{k+1}-w^k\|^2\\
\le&\|x^k-p\|^2-\left[1-\left(\theta_k+\gamma_k\frac{\nu}{\gamma_{k+1}}\right)^2\right]\|y^k-x^k\|^2\\
&+2(\gamma_0+M)\nu_k+2\varrho_k\langle x^0-p,w^k-p\rangle\\
&-\|x^{k+1}-w^k\|^2\\
\le&\|x^k-p\|^2-\left[1-\left(\theta_k+\gamma_k\frac{\nu}{\gamma_{k+1}}\right)^2\right]\|y^k-x^k\|^2\\
&+2(\gamma_0+M)\nu_k+2+\varrho_kM_1-\|x^{k+1}-w^k\|^2,
\end{aligned}\tag{9}$$

where $M_1:=\sup\{2\langle x^0-p,w^k-p\rangle : k=0,1,...\}<\infty$, which implies that

$$\begin{aligned}
&\left[1-\left(\theta_k+\gamma_k\frac{\nu}{\gamma_{k+1}}\right)^2\right]\|y^k-x^k\|^2+\|x^{k+1}-w^k\|^2\\
&\le-\xi_{k+1}+\xi_k++2(\gamma_0+M)\nu_k+2+\varrho_kM_1,
\end{aligned}$$

for every $k\ge K_0$. Taking the limit as $k\to\infty$ on both sides of the last inequality and using Lemma 3.1 (i) and Condition (3), we obtain

$$\lim_{k\to\infty}\|x^{k+1}-w^k\|=\lim_{k\to\infty}\|x^k-y^k\|=0. \tag{10}$$

Observe that

$$\begin{aligned}
\|z^k-y^k\|&=\|\theta_ky^k-\theta_kx^k+\gamma_k(u^k-v^k)-x^k\|\\
&\le\theta_k\|y^k-x^k\|+\gamma_k\frac{\nu}{\gamma_{k+1}}\|y^k-x^k\|\\
&=\left(\theta_k+\gamma_k\frac{\nu}{\gamma_{k+1}}\right)\|y^k-x^k\|,
\end{aligned}$$

which together with (10) implies that $\lim_{k\to\infty}\|z^k-y^k\|=0$. By the definition of x^{k+1} and boundedness of the sequence $\{z^k\}$, we have

$$\|w^k-z^k\|=\varrho_k\|x^0-z^k\|\le\varrho_kQ_1\to0\ as\ k\to\infty,$$

where $Q_1=\sup\{\|x^0-z^k\|:\ k=0,1,...\}<+\infty$. This together with $\lim_{k\to\infty}\|z^k-x^k\|=0$ implies that

$$\|w^k-x^k\|\le\|w^k-z^k\|+\|z^k-x^k\|\to0\ as\ k\to\infty. \tag{11}$$

By the definition of w^k in **Step 3** and the inequality $\|u+v\|^2\le\|u\|^2+2\langle v,u+v\rangle\quad\forall u,v\in\mathbb{H}$, we get

$$\begin{aligned}
&\|w^k-p\|^2=\|\varrho_k(x^0-p)+(1-\varrho_k)(z^k-p)\|^2\\
\le&(1-\varrho_k)^2\|z^k-p\|^2+2\varrho_k(1-\varrho_k)\langle x^0-p,x^{k+1}-p\rangle\\
\le&(1-\varrho_k)\|z^k-p\|^2+2\varrho_k\langle x^0-p,w^k-p\rangle.
\end{aligned}$$

From the last inequality and Lemma 3.1 (ii), it follows that

$$\begin{aligned}
&\|x^{k+1}-p\|^2\le(1-\varrho_k)\|x^k-p\|^2\\
&+2\varrho_k\langle x^0-p,x^{k+1}-p\rangle+(1-\varrho_k)[2(\gamma_0+M)\nu_k]\\
\le&(1-\varrho_k)\|x^k-p\|^2+2\varrho_k\langle x^0-p,w^k-p\rangle\\
&+2(\gamma_0+M)\nu_k,
\end{aligned}$$

which implies that

$$\xi_{k+1}\le(1-\varrho_k)\xi_k+\varrho_k\alpha_k+\beta_k. \tag{12}$$

On the other hand, since the sequence $\{w^k\}$ is bounded, there exists a subsequence $\{w^{k_i}\}$ such that $w^{k_i}\rightharpoonup z$ as $i\to\infty$ and

$$\limsup_{k\to\infty}\langle x^0-p,w^k-p\rangle=\lim_{i\to\infty}\langle x^0-p,w^{k_i}-p\rangle. \tag{13}$$

We deduce from (11) that $x^{k_i}\rightharpoonup z$ as $i\to\infty$. Applying $\lim_{k\to\infty}\|x^{k+1}-w^k\|=\lim_{k\to\infty}\|z^k-y^k\|=\lim_{k\to\infty}\|x^k-y^k\|=0$ and Lemma 3.2, we get $z\in\Omega$. From this, (13) and Lemma 2.1 (i), it follows that

$$\limsup_{k\to\infty}\alpha_k=2\langle x^0-p,z-p\rangle\le0.$$

By using Lemma 2.4, the last inequality, $\limsup_{k\to\infty}\alpha_k\le0$ and Condition (3), we deduce

$$\lim_{k\to\infty}\xi_k=\lim_{k\to\infty}\|x^k-p\|^2=0.$$

Thus, $\{x^k\}$ converges strongly to the solution $p=P_\Omega(x^0)$.

Case 2. We now assume that there is not $\bar{k}\in\mathbb{N}$ such that $\{\xi_k\}_{k=\bar{k}}^{\infty}$ is monotonically decreasing. Then, there exists an integer $k_0\ge\bar{k}$ such that $\xi_{k_0}\le\xi_{k_0+1}$. We have from Lemma 2.5 that there exists a subsequence $\{\xi_{\tau(k)}\}$ of $\{\xi_k\}$ such that

$$0\le\xi_k\le\xi_{\tau(k)+1},\xi_{\tau(k)}\le\xi_{\tau(k)+1}\quad\forall k\ge k_0,$$

where $\tau(k)=\max\{i\in\mathbb{N}:k_0\le i\le k,\xi_i\le\xi_{i+1}\}$. Using $\xi_{\tau(k)}\le\xi_{\tau(k)+1},\ \forall k\ge k_0$ and (9), one has

$$\begin{aligned}
0\le&\left[1-\left(\theta_{\tau(k)}+\gamma_{\tau(k)}\frac{\nu}{\gamma_{\tau(k)+1}}\right)^2\right]\|y^{\tau(k)}-x^{\tau(k)}\|\\
&+\|x^{\tau(k)+1}-w^{\tau(k)}\|^2\\
&\le-\xi_{\tau(k)+1}+\xi_{\tau(k)}+\varrho_{\tau(k)}M_1+2(\gamma_0+M)\nu_{\tau(k)}\\
&\le\varrho_{\tau(k)}M_1+2(\gamma_0+M)\nu_{\tau(k)}.
\end{aligned}$$

Passing to the limit in the above inequa and taking into account Condition (3), we obtain $\lim_{k\to\infty}\|y^{\tau(k)} - x^{\tau(k)}\| = \lim_{k\to\infty}\|x^{\tau(k)+1} - w^{\tau(k)}\|^2 = 0$. By the same arguments as in the *Case 1*, we can show that

$$\lim_{n\to\infty}\|x^{\tau(k)+1} - x^{\tau(k)}\| = \lim_{n\to\infty}\|x^{\tau(k)} - z^{\tau(k)}\| = \lim_{n\to\infty}\|z^{\tau(k)} - y^{\tau(k)}\| = 0. \tag{14}$$

Since $\{w^{\tau(k)}\}$ is bounded, there exists a subsequence of $\{w^{\tau(k)}\}$, still denoted by $\{w^{\tau(k)}\}$, which converges weakly to z. Following similar arguments as in *Case 1*, we conclude that $z \in \Omega$ and

$$\limsup_{k\to\infty}\alpha_{\tau(k)} \le 0. \tag{15}$$

We deduce from (12) and $\xi_{\tau(k)} \le \xi_{\tau(k)+1},\ \forall k \ge k_0$ that

$$\begin{aligned}\varrho_{\tau(k)}\xi_{\tau(k)} &\le \xi_{\tau(k)} - \xi_{\tau(k)+1} + \varrho_{\tau(k)}\alpha_{\tau(k)} + \beta_{\tau(k)} \\ &\le \varrho_{\tau(k)}\alpha_{\tau(k)} + \beta_{\tau(k)}.\end{aligned}$$

It is equivalent to $\xi_{\tau(k)} \le \alpha_{\tau(k)} + \frac{\beta_{\tau(k)}}{\varrho_{\tau(k)}}$. From (15), Condition (3) and the last inequality, it follows that

$$\limsup_{k\to\infty}\xi_{\tau(k)} \le \limsup_{k\to\infty}\alpha_{\tau(k)} \le 0,$$

which implies that $\lim_{k\to\infty}\xi_{\tau(k)} = 0$. We have

$$\begin{aligned}\sqrt{\xi_{\tau(k)+1}} &= \|x^{\tau(k)+1} - p\| \\ &\le \|x^{\tau(k)+1} - x^{\tau(k)}\| + \|x^{\tau(k)} - p\| \\ &\le \|x^{\tau(k)+1} - x^{\tau(k)}\| + \xi_{\tau(k)}.\end{aligned}$$

Taking the limit as $k \to \infty$ on both sides of the last inequality and using (14), we obtain $\lim_{k\to\infty}\xi_{\tau(k)+1} = 0$. This together with $0 \le \xi_k \le \xi_{\tau(k)+1}$ for all $k \ge k_0$ implies that $\lim_{k\to\infty}\xi_k = 0$. It means that the sequence $\{x^k\}$ converges strongly to $p \in \Omega$. The proof is complete. $\square$

As an illustration, we apply Algorithm 3.1. to solve the well-known Nash-Cournot oligopolistic market equilibrium model with equilibrium constraint in [9]. Consider a class of well-known problem oligopolistic market equilibrium problem Nash-Cournot between n firms in the space $\mathbb{R}^n$. Let $g_j(x_j)$, $p_j(\delta_x)$ and $f_j(x)$ denote respectively the total cost, the price function and the profit function of firm j, where the quantity of product $\delta_x := x_1 + x_2 + ... + x_n$. Then, we have $f_j(x) = f_j(x_1, x_2, ..., x_n) = x_j p_j(\delta_x) - g_j(x_j)$. Let C_j be any set of firm j consisting of its possible production levels, C_j be nonempty, bounded and $x_j \in C_j$. Set $C = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \ x_j \in C_j\}$ and

$$f(x,y) = \langle F(x), y - x\rangle + g(y) - g(x),$$

where $g(x) = \sum_{i=1}^n g_j(x_j)$, $F(x) = (F_1(x), F_2(x), ..., F_n(x))$, $F_j(x) = -p_j(\delta_x) - x_j p_j'(\delta_x)$. Then, the Nash equilibrium can be rewritten as the following equilibrium problem: Find $p \in C$ such that

$$f(p,x) \ge 0, \quad \forall x \in C.$$

Obviously, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is pseudomonotone, L-Lipschitz continuous and $g : \mathbb{R}^n \to \mathbb{R}$ is convex, differentiable then the function $f(x,y)$ satisfies assumptions A_1, A_2, A_4. We now consider the case that the function $p_j(\delta_x)$ is affine

$$p_j(\delta_x) = \alpha_j - \beta_j\delta_x, \ \ \beta_j \ge 0, \alpha_j \ge 0, \quad \forall j = 1, ..., n.$$

Then

$$\begin{aligned}F_j(x) &= -p_j(\delta_x) - x_j p_j'(\delta_x) = \beta_j\delta_x - \alpha_j + \beta_j x_j \\ &= 2\beta_j x_j + \beta_j\sum_{j=1, j\ne i}^n x_j - \alpha_j,\end{aligned}$$

and so, $F(x) = Bx - \alpha$, where

$$B = \begin{pmatrix} 2\beta_1 & \beta_1 & ... & \beta_1 \\ \beta_2 & 2\beta_2 & ... & \beta_2 \\ ... & & & \\ \beta_n & \beta_n & ... & 2\beta_n \end{pmatrix}, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T.$$

It is known that B is a positive symmetric matrix and F is monotone and $\|B\|$-Lipschitz continuous ([9]). Therefore, this model can be solve by Algorithm 3.1.

4 CONCLUSIONS

We have introduced a new projection algorithm for finding a common point of the solution set of Problem (EPs) and the set of fixed points of a demicontractive mappings. Our algorithm only uses one projection on C at each iteration. We show that the proposed algorithm is strongly convergent under the mild assumptions. We also apply the proposed algorithms to solve a oligopolistic Nash-Cournot equilibrium model.

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