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ON THE STABILITY OF IMPLICIT INTEGRO-DYNAMIC EQUATION ON TIME SCALES

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Abstract:

This paper deals with the robust stability of implicit integro - dynamic equations. We consider the solvability of the equation and then the preservation of exponential stability under small perturbations.

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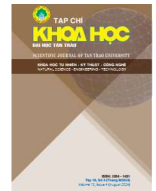

VỀ TÍNH ỔN ĐỊNH CỦA PHƯƠNG TRÌNH ĐỘNG HỌC TÍCH PHÂN ẨN TRÊN THANG THỜI GIAN

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Abstract:

Trong bài báo này, chúng ta trình bày bài toán về tính ổn định của phương trình động học tích phân ẩn trên thang thời gian. Cụ thể, chúng ta xét tính giải được của phương trình và chứng minh được rằng, dưới tác động của nhiễu, nghiệm của những phương trình này bảo toàn tính bị chặn và tính ổn định mũ.

1 Introduction

The theory of implicit integro-dynamic equations has found many applications in demography, the study of materials, and in actuarial science through the renewal equation (A. S. Andreev *et al* 2018; H. Brunner 2017;Yu, L., Daleckii *et al* 1971). However, relatively few kinds of implicit integro-dynamic equations and systems can be solved explicitly. Therefore, during scientific investigations, researchers need to find the methods which allow them to study the qualitative behavior of their solutions without solving them. One of important problems in studying the qualitative theory is to investigate the robust stability of systems. The robust stability is considered for difference singular equations or dynamic equations on time scales in (Du, N. H. *et al* 2016; D.D. Thuan *et al* 2019), although all most works consider only systems without or finite memory. Therefore, it is worth considering the robust stability of implicit integro - dynamic equations on time scales. The aim of this paper is to continue the study of this problem by considering the robust stability for the implicit integro-dynamic system on time scales under the form

$$A(t)x^{\Delta}(t) = B(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + f(t)$$

with $t \geq t_0$ and $A(\cdot), B(\cdot), K(\cdot,\cdot), f(\cdot)$ are specified later. We deal with the preservation of the stability for this dynamic equation under small perturbations. Since the derivative of state process $x(t)$ at time t depends on all past path $x(s), t_0 \leq s \leq t$, we have to use a more general inequality of Gronwall-Bellman type to obtain the upper bound of perturbations.

The paper is organized as follows. In the next section we recall some basic notions and preliminary results on time scales. In section 3, we consider the solvability of implicit integro - dynamic equations. Finally, in section 4, we are concerned with conditions such that if the solution of a implicit integro - dynamic equations is uniformly stable/exponentially stable, then under small Lipschitz perturbations it is still uniformly stable/exponentially stable.

2 Premilinary

2.1 Time scales

A time scale is a nonempty closed subset of the real numbers, enclosed with the topology inherited from the standard topology on $\mathbb{R}$. We usually denote it by the symbol $\mathbb{T}$. On the time scale $\mathbb{T}$, we define the forward jump operator $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the graininess $\mu(t) = \sigma(t) - t$. Similary, the backward operator is defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ and the backward graininess is $\mu(t) = t - \rho(t)$.

A regulated function f is called *rd-continuous* if it is there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point and continuous at every right-dense point. The set of rd-continuous functions defined on the interval J valued in X will be denoted by $C_{rd}(J, X)$. A function f from $\mathbb{T}$ to $\mathbb{R}$ is *regressive* (resp., *positively regressive*) if for every $t \in \mathbb{T}$, then $1+\mu(t)f(t) \neq 0$ (resp., $1+\mu(t)f(t) > 0$). We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp., $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$) the set of (resp., positively regressive) regressive functions, and $C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp., $C_{rd}\mathcal{R}^+(\mathbb{T}, \mathbb{R})$) the set of rd-continuous (resp., positively regressive) regressive functions from $\mathbb{T}$ to $\mathbb{R}$. For all $x, y \in \mathbb{T}$, we define the *circle plus* and he *circle minus*:

$$p \oplus q := p + q + \mu(t)xy, \qquad p \ominus q := \frac{p-q}{1+\mu(t)q}.$$

It is easy to verify that, for all $p, q \in \mathcal{R}$, $p \oplus q, p \ominus q, \ominus p, \ominus q \in \mathcal{R}$. Element $(\ominus q)(\cdot)$ is called the inverse element of element $q(\cdot) \in \mathcal{R}$. Hence, the set $\mathcal{R}(\mathbb{T}, \mathbb{R})$ with the calculation $\oplus$ forms an Abelian group.

Definition 2.1 (Delta Derivative). *A function $\varphi : \mathbb{T} \to R^d$ is called delta differentiable at t if there exists a vector $\varphi^{\Delta}(t)$ such that for all $\varepsilon > 0$,*

$$\|\varphi(\sigma(t)) - \varphi(s) - \varphi^{\Delta}(t)(\sigma(t) - s)\| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in (t-\delta, t+\delta) \cap T$ and for some $\delta > 0$. The vector $\varphi^{\Delta}(t)$ is called the delta derivative of f at t.

2.2 Exponential Functions

Let $\mathbb{T}$ be an unbounded above time scale, that is $\sup \mathbb{T} = \infty$.

Definition 2.2 (Exponential stability function). *Let $p : \mathbb{T} \to \mathbb{R}$ is regressive, we define the exponential function by*

$$e_p(t, t_0) = \exp\Big\{ \int_{t_0}^{t} \lim_{h \searrow \mu(s)} \frac{\text{Ln}(1 + hp(s))}{h} \Delta s \Big\}.$$

Properties of the exponential function: If p, q are regressive, rd-continuous functions and

$t, r, s \in \mathbb{T}$ then the following hold:

$$e_p(t,s)e_q(t,s) = e_{p+q}(t,s).$$
$$e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s);$$
$$e_p(t,s)e_p(s,r) = e_p(t,r).$$

Theorem 2.3 (see (Bohner, M. *et al* 2001)). *If p is regressive and $t_0 \in \mathbb{T}$, then $e_p(.,t_0)$ is a unique solution of the initial value problem*

$$x^{\Delta}(t) = p(t)x(t),\ x(t_0) = 1.$$

Let $\mathbb{T}$ be time scale that is unbounded above. For any $a, b \in \mathbb{R}$, the notation $[a,b]$ or (a,b) means the segment on $\mathbb{T}$, that is $[a,b] = \{t \in \mathbb{T} : a \le t \le b\}$ or $(a,b) = \{t \in \mathbb{T} : a < t < b\}$ and $\mathbb{T}_a = \{t \ge a : t \in \mathbb{T}\}$. We can define a measure $\Delta_{\mathbb{T}}$ on $\mathbb{T}$ by considering the Caratheodory construction of measures when we put $\Delta_{\mathbb{T}}[a,b) = b-a$. The Lebesgue integral of a measurable function f with respect to $\Delta_{\mathbb{T}}$ is denoted by $\int_a^b f(s)\Delta_{\mathbb{T}}s$ (see (Guseinov, G. Sh. 2003)).

2.3 Some surveys on linear algebra

We survey briefly some basic properties of linear implicit dynamic equation.

Lemma 2.4. *Let A and B be given $n \times n$ matrices, and Q be a projector onto $Ker A$, i.e., $Q^2 = Q, Im Q = Ker A$. Denote $S = \{x : Bx \in Im A_\sigma\}$. Let T be a continuous function defined on $\mathbb{T}_a$, taking values in $Gl(\mathbb{R}^n)$ such that $T|Ker A_\sigma$ is an isomorphism between $Ker A_\sigma$ and $Ker A$. The following assertions are equivalent*

a) $S \cap Ker A = \{0\}$.

b) $G = A_\sigma - BT Q_\sigma$ *is nonsingular.*

c) $\mathbb{R}^n = S \oplus Ker A$.

Proof. The proof of this lemma can be found in (R. März. 1998), Appendix 1, Lemma A1, p.329. □

Lemma 2.5. *$A, B,\ Q, G$ mentioned in Lemma 2.4 and suppose that the matrix G is nonsingular. Then, there hold the following relations:*

a) $P_\sigma = G^{-1} A_\sigma$ *where* $P_\sigma = I - Q_\sigma$.

b) $-G^{-1} B T Q_\sigma = Q_\sigma$.

c) $\hat{Q} := -TQ_\sigma G^{-1} B$, *called canonical projector, is the projector onto $Ker A$ along S.*

d) $TQ_\sigma G^{-1}$ *does not depend on the choice of T and Q.*

Proof. The results in this lemma are proved in (R. März. 1998), p.319. □

To consider the robust stability we need the Gronwall-Bellman's inequality. It will be introduced and applied in the following lemma.

Lemma 2.6. *(see (Choi, S. K. et al 2010)). Let the functions $u(t), \sigma(t), v(t),\ w(t,r)$ be nonnegative and continuous for $a \le \tau \le r \le t$, and let c_1 and c_2 be nonnegative. If for $t \in \mathbb{T}_a$*

$$u(t) \le \varphi(t)\left[c_1 + c_2\int_\tau^t \left[v(s)u(s) + \int_\tau^s w(s,r)u(r)\Delta r\right]\Delta s\right]$$

then with $p(\cdot) = c_2\left[v(\cdot) + \int_\tau^{\cdot} w(\cdot,r)\Delta r\right]$,

$$u(t) \le c_1\varphi(t)e_{p(\cdot)}(t,\tau), \quad t \ge \tau.$$

3 Solvability of implicit integro-dynamic equations

Let $A(\cdot), B(\cdot)$ be two continuous functions defined on $\mathbb{T}_{t_0}$, valued in the set of $n \times n-$ matrices ($\mathbb{R}^{n\times n}$ for brief), $f \in L_p^{loc}(\mathbb{T}_{t_0}; \mathbb{R}^n)$

and $K(\cdot,\cdot)$ be a two-variable continuous function defined on the set $\{(t,s) : t_0 \le s \le t < \infty\}$, valued in $\mathbb{R}^{n\times n}$. Consider the linear implicit dynamic equations on time scales (IDE for short)

$$A_\sigma(t)x^\Delta(t) = B(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + f(t) \tag{3.1}$$

To solve this equation, we suppose that $\operatorname{Ker} A(\cdot)$ is smooth in the sense there exists a continuously Δ–differentiable projector $Q(t)$ onto $\operatorname{Ker} A(t)$, i.e., Q is continuously differentiable and $Q^2 = Q$, $\operatorname{Im} Q(t) = \operatorname{Ker} A(t)$ for all $t \in \mathbb{T}_{t_0}$. By setting $P = I - Q$ we can rewrite the equation (3.1) as

$$A_\sigma(t)\big(Px\big)^\Delta(t) = \overline{B}(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + f(t), \tag{3.2}$$

where $\overline{B} := B + A_\sigma P^\Delta$. It is seen that the solution $x(\cdot)$ of the equation (3.2), if it exists, is not necessarily differentiable but it is required that the component $Px(\cdot)$ is Δ–differentiable almost everywhere on $\mathbb{T}_{t_0}$.

Consider the space $\mathcal{C}^1_A\big(\mathbb{T}_{t_0};\mathbb{R}^n\big)$ is the set of $y \in C\big(\mathbb{T}_{t_0};\mathbb{R}^n\big)$ such that $Py(\cdot)$ is almost everywhere- differentiable on $\mathbb{T}_{t_0}$.

Define the linear operators $G := A_\sigma - \overline{B}TQ_\sigma$. It is clear that $G \in L^{loc}_\infty\big(\mathbb{T}_{t_0};\mathbb{R}^{n\times n}\big)$.

Definition 3.1. *The IDE* (3.1) *is said to be index-1 if $G(t)$ is invertible for all $t \in \mathbb{T}_{t_0}$.*

For any $T > t_0$, consider two subspaces:

$$C_{TQ_\sigma G^{-1}([t_0,T);\mathbb{R}^n)} = \{v \in C([t_0,T);\mathbb{R}^n) : v(t) \in \operatorname{Im} TQ_\sigma G^{-1}(t)\},$$
$$\mathcal{C}_P([t_0,T);\mathbb{R}^n) = \{u \in C([t_0,T);\mathbb{R}^n) : u(t) \in \operatorname{Im} P(t)\}.$$

Theorem 3.2. *For any $t_0 \ge 0$ and $x_0 \in \mathbb{R}^n$, the equation* (3.2) *has a unique solution $x(\cdot) \in \mathcal{C}^1_A\big(\mathbb{T}_{t_0};\mathbb{R}^n\big)$, satisfying the initial condition*

$$P(t_0)(x(t_0) - x_0) = 0. \tag{3.3}$$

Proof. We divide the proof of Theorem into steps.

• Split the solution $x(\cdot)$ into $Px(\cdot) + Qx(\cdot)$ and try to solve $u(\cdot) = Px(\cdot)$ and $v(\cdot) = Qx(\cdot)$. Multiplying both sides of (3.2) with $P_\sigma G^{-1}$, $Q_\sigma G^{-1}$ and using the properties

$$P_\sigma = G^{-1} A_\sigma, \qquad -G^{-1}\,\overline{B}\,TQ_\sigma = Q_\sigma,$$

we obtain, for $t \ge t_0$,

$$u^\Delta(t) = \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)(t)u(t) + P_\sigma G^{-1} f(t) + P_\sigma G^{-1}\int_{t_0}^{t} K(t,s)\big(u(s)+v(s)\big)\Delta s, \tag{3.4}$$

$$v(t) = TQ_\sigma G^{-1}\overline{B}u(t) + TQ_\sigma G^{-1} f(t) + TQ_\sigma G^{-1}\int_{t_0}^{t} K(t,s)\big(u(s)+v(s)\big)\Delta s, \tag{3.5}$$

• Consider the equation (3.5) by defining an operator $H : C\big([t_0,\infty);\mathbb{R}^n\big) \to C\big([t_0,\infty);\mathbb{R}^n\big)$ given by

$$(Hv)(t) = v(t) - TQ_\sigma G^{-1}\int_{t_0}^{t} K(t,s)v(s)\Delta s. \tag{3.6}$$

From [?, Theorem 3.1] it follows that the continuity of $TQ_\sigma G^{-1}(\cdot)K(\cdot,\cdot)$ implies the invertibility of H because $(Hv)(t) = y(t), t \ge t_0$ is a Volterra integral equation of second kind. Precisely,

$$(H^{-1}y)(t) = y(t) + \sum_{n=1}^{\infty}\int_{t_0}^{t} K_n(t,s)y(s)\Delta s \tag{3.7}$$

where, K_n is defined by induction

$$K_1(t,s) = TQ_\sigma G^{-1}(t)K(t,s),$$
$$K_{n+1}(t,s) = \int_s^t K_n(t,\tau)TQ_\sigma G^{-1}(\tau)K(\tau,s)\Delta\tau,$$

for $t \ge s \ge t_0,\ n \ge 1$. Paying attention that for any $T > t_0$ the following inequality holds

$$\sup_{t_0 \le s \le t \le T} \|K_n(t,s)\| \le \Big(\sup_{t_0 \le s \le t \le T} \big\|TQ_\sigma G^{-1}(t)K(t,s)\big\|\Big)^n \frac{(T-t_0)^n}{n!},$$

which implies that the serie s

$$R(t,s) = I + \sum_{n=1}^{\infty} K_n(t,s).$$

is uniformly convergent on the set $\{(t,s)\}$ such that $t_0 \le s \le t \le T$ and $R(\cdot,\cdot)$ is continuous. Thus, H^{-1} is also a second kind linear Volterra operator with the kernel $R(\cdot,\cdot)$. This means that H is a continuous bijection on $C([t_0,T];\mathbb{R}^n)$.

- We now try to simplify the form of (3.5). From this equation we get

$$v(t) = H^{-1}TQ_\sigma G^{-1}\Big[\overline{B}u + \int_{t_0}^{\cdot} K(\cdot,s)u(s)ds\Big](t) + (H^{-1}TQ_\sigma G^{-1}f)(t). \tag{3.8}$$

In noting that

$$H^{-1}TQ_\sigma G^{-1}\left(\int_{t_0}^{\cdot} K(\cdot,s)u(s)ds\right)(t) = H^{-1}\big(u - Hu\big)(t) = (H^{-1}u)(t) - u(t),$$

we can rewrite (3.8) as

$$v(t)=(H^{-1}\hat{P}u)(t) - u(t)+(H^{-1}Q_\sigma G^{-1}f)(t) \tag{3.9}$$

where $\hat{Q}(t) = I - \hat{P}(t) = -TQ_\sigma G^{-1}\overline{B}(t)$ is the canonical projector onto $\operatorname{Ker} A(t)$.

- Substituting $v(t)$ into (3.4) obtains

$$u^\Delta(t)= \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)u(t) + P_\sigma G^{-1}f(t) + P_\sigma G^{-1}\int_{t_0}^{t} K(t,s)H^{-1}\big(\hat{P}u + TQ_\sigma G^{-1}f\big)(s)\Delta s$$

for all $t \ge t_0$. (3.10)

We now use the following lemma, its proof can be easily obtained by using Picard's approximation method and usual procedures.

Lemma 3.3. *Let S be a function defined on $[t_0,T] \times C_P\big([t_0,T];\mathbb{R}^n\big)$, valued in $\mathbb{R}^n$, such that $S(t,u)$ depends only the values of u on $[t_0,t]$ for every $u \in C_P\big([t_0,T];\mathbb{R}^n\big)$ and S satisfies the Lipschitz condition, i.e., there is a constant $k > 0$ such that*

$$\|S(t,y_1) - S(t,y_2)\| \le k \sup_{t_0 \le s \le t} \|y_1(s) - y_2(s)\|,$$

for all $t \in [t_0,T]$, $y_1, y_2 \in C_P\big([t_0,T];\mathbb{R}^n\big)$. Then the equation

$$y^\Delta=(P^\Delta+P_\sigma G^{-1}\overline{B})y + P_\sigma G^{-1}S(t,y), \tag{3.11}$$

with the initial condition $y(t_0) = P(t_0)x_0$ has a unique solution in $C_P([t_0,T];\mathbb{R}^n)$. Moreover, there exists a constant c such that if $y(t)$ and $z(t)$ are two solutions of (3.11) then

$$\|y(t) - z(t)\| \le c\,\|y(t_0) - z(t_0)\|, \tag{3.12}$$

By using this lemma, we see that the equation (3.10) has a unique solution $u(\cdot)$ with initial condition $u(t_0) = P(t_0)x_0$. Then, we use the formula (3.9) to obtain the solution of (3.2) as

$$x(t) = u(t) + v(t) =(H^{-1}\hat{P}u)(t) + (H^{-1}TQ_\sigma G^{-1}f)(t), \tag{3.13}$$

for $t \ge t_0$. The proof is complete. □

Remark 3.4. *i) Inspired by the above decoupling procedure, we state the initial condition $u(t_0) = P(t_0)x_0$, or equivalent to*

$$P(t_0)(x(t_0) - x_0) = 0,\ x_0 \in \mathbb{R}^n. \tag{3.14}$$

We note that the condition (3.14) does not depend on the chosen projector opertor $Q(t_0)$.
ii) Let $u(t)$ be the solution of the equation (3.10). Multiplying both sides of this equation with Q_σ yields $Q_\sigma u^\Delta = Q_\sigma P^\Delta u$. Paying attention that $Q^\Delta = (Q^2)^\Delta = Q^\Delta Q_\sigma + QQ^\Delta$ obtains

$$(Qu)^\Delta = Q^\Delta Qu$$

Thus, if $Q(t_0)u(t_0) = 0$ then $Q(t)u(t) = 0$ for all $t \ge t_0$. This means that (3.10) has the invariant property: every solution starting in

$Im\,P(t_0)$ remains in $Im\,P(t)$ for all $t > t_0$ $x(t_0) \in Im\,P(t_0)$ then $x(t) \in Im\,P(t)$, for all $t \in \mathbb{T}_{t_0}$.

ii) Since $TQ_\sigma G^{-1}$ is independent of the choice of Q, so is the operator H.

iii) We note that for every $T > t_0$, the space $C_{Q_\sigma G^{-1}}([t_0, T]; \mathbb{R}^n)$ is independent of the choice Q_σ and it is invariant under the the operator H.

We now try to give the variation of constants formula for the solution $x(\cdot)$ of the equation (3.2). In order to do that, first we consider the homogeneous equation, i.e., $f = 0$

$$A_\sigma\big(Py\big)^\Delta(t) = \overline{B}y(t) + \int_{t_0}^{t} K(t,s)y(s)\Delta s. \quad (3.15)$$

Define the Cauchy matrix $\Phi(t,s), t \geq s \geq t_0$ generated by homogeneous system (3.15) as the solution of the equation

$$A(t)\Phi^\Delta(t,s) = B(t)\Phi(t,s) + \int_{s}^{t} K(t,\tau)\Phi(\tau,s)\Delta\tau,$$

and $P(s)\,(\Phi(s,s) - I) = 0$.Then, we have the variation of constants formula for the solution of (3.2)

Theorem 3.5. *The solution $x(\cdot)$ of the equation* (3.2) *with the initial condition $P(t_0)(x(t_0) - x_0) = 0$ can be expressed as*

$$x(t) = \Phi(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)P_\sigma G^{-1}(\tau)$$
$$\Big(f(\tau) + \int_{t_0}^{\tau} K(\tau,h)(H^{-1}TQ_\sigma G^{-1}f)(h)\Delta h\Big)\Delta\tau$$
$$+ (H^{-1}TQ_\sigma G^{-1}f)(t),\ t \geq t_0. \quad (3.16)$$

Proof. A similar procedure to split the solution of the homogeneous equation (3.15) into $y(\cdot) = \overline{u}(\cdot) + \overline{v}(\cdot)$ obtains

$$\overline{u}^\Delta(t) = \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)\overline{u}(t)$$
$$+ P_\sigma G^{-1}\int_{t_0}^{t} K(t,s)\big(H^{-1}\hat{P}\overline{u}\big)(s)\Delta s, \quad (3.17)$$

And $y(t) = (H^{-1}\hat{P}\overline{u})(t)$. (3.18)

Denote by $\Phi_0(\cdot,\cdot)$ the Cauchy operator of (3.17), i.e., it is the solution of the matrix equation

$$\Phi_0^\Delta(t,s) = \big(P^\Delta + P_\sigma G^{-1}\overline{B}\big)\Phi_0(t,s)$$
$$+ P_\sigma G^{-1}(t)\int_{s}^{t} K(t,\tau)\big(H^{-1}\hat{P}_\sigma\Phi_0(\cdot,s)\big)(\tau)\Delta\tau$$

and $\Phi_0(s,s) = I$ for all $t \geq s \geq t_0$. Then, by directly differentiating both sides we obtain the variation constants formula for the solution $u(\cdot)$ of (3.10) with the initial condition $u(t_0) = P(t_0)x_0$

$$u(t) = \Phi_0(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi_0(t,\tau)P_\sigma G^{-1}\Big(f(\tau)$$
$$+ \int_{t_0}^{\tau} K(\tau,h)H^{-1}TQ_\sigma G^{-1}f(h)\Delta h\Big)\Delta\tau. \quad (3.19)$$

On the other hand, since $\overline{u}(t) = \Phi_0(t,t_0)P(t_0)x_0$ and by (3.18) we have the relation between $\Phi(t,s)$ and $\Phi_0(t,s)$

$$\Phi(t,s) = \big(H^{-1}\hat{P}\,\Phi_0(\cdot,s)P(s)\big)(t). \quad (3.20)$$

Thus, by acting $H^{-1}\hat{P}$ to both sides of (3.19) and paying attention to the expression (3.13) it is seen that the unique solution $x(\cdot)$ of (3.2) with the initial condition $P(t_0)(x(t_0) - x_0) = 0$ can be given by the variation of constants formula (3.16). The proof is complete. □

Assumption 3.6. *There exists a differentiable projector $Q_\sigma(\cdot)$ onto $Ker\,A(\cdot)$ such that $TQ_\sigma G^{-1}$ and $P = I - Q$ are bounded on $[t_0, \infty)$.*

Definition 3.7. i) *The implicit integro - dynamic equation* (3.15) *is uniformly stable if and only if there exists a positive number $M_0 > 0$ such that*

$$\|\Phi(t,s)\| \leq M_0,\ t \geq s. \quad (3.21)$$

ii) *Let ω is regressive projective. The integro - equation* (3.15) *is said to be ω-exponentially stable if and only if there exists a positive number M such that*

$$\|\Phi(t,s)\| \leq Me_{\ominus\omega}(t,s),\ t \geq s. \quad (3.22)$$

4 Stability of implicit integro - dynamic equation under small perturbations

In this section, we consider the effect of small nonlinear perturbations to the stability of implicit integro - equation (3.15). Suppose that for every $t \geq t_0$, the perturbed equation has the form

$$A_\sigma(t)x^\Delta(t) = B(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + F(t,x(t)), \;\; t \in \mathbb{T}_{t_0}. \tag{4.1}$$

Assume that $F(t,0) = 0$, for all $t \geq t_0$, which follows that the equation (4.1) has the trivial solution $x(\cdot) \equiv 0$. First at all, we consider the solvability of (4.1).

Assumption 4.1. *For all $t \geq t_0$, the functions $P_\sigma G^{-1}(t)F(t,x)$ and $TQ_\sigma G^{-1}(t)F(t,x)$ are Lipschitz in x with Lipschitz coefficient l_t and γ_t respectively. Suppose further that $l.$ and $\gamma.$ are continuous functions.*

We endow $C_{TQ_\sigma G^{-1}}([t_0,T];\mathbb{R}^n)$ with the norm inherited from $C([t_0,T];\mathbb{R}^n)$ and understand that $\|H^{-1}\|$ mean that the norm of operator H^{-1} in $C_{TQ_\sigma G^{-1}}([t_0,T];\mathbb{R}^n)$. By denoting $\overline{\gamma}_t = \sup_{t_0 \leq s \leq t} \gamma_s$ for $t \geq t_0$, we have

Lemma 4.2. *Let $T > t_0$. If $\overline{\gamma}_T \|H^{-1}\| < 1$, then the equation (4.1) with the initial condition*

$$P(t_0)(x(t_0) - x_0) = 0, \tag{4.2}$$

is solvable on $[t_0,T]$. Further, there exists a constant M_T such that

$$\|x(t)\| \leq M_T \|P(t_0)x(t_0)\|, \text{ for all } t_0 \leq t \leq T.$$

Proof. As in the proof of Theorem 3.2, denoting $u(\cdot) = Px(\cdot)$ and $v(\cdot) = Qx(\cdot)$ comes to

$$u^\Delta(t) = \left(P^\Delta + P_\sigma G^{-1}\overline{B}\right)u(t) + P_\sigma G^{-1}\int_{t_0}^{t} K(t,s)H^{-1}\left(\hat{P}u + TQ_\sigma G^{-1}f\right)(s)\Delta s + P_\sigma G^{-1}F\big(t,x(t)\big) \tag{4.3}$$

for $T \geq t \geq t_0$. And

$$x(t) = u(t) + v(t) = (H^{-1}\hat{P}u)(t) + \left(H^{-1}TQ_\sigma G^{-1}F\big(\cdot,x(\cdot)\big)\right)(t),$$

for $T \geq t \geq t_0$. Fix $u(\cdot) \in C_P\big([t_0,T];\mathbb{R}^n\big)$ and consider the mapping $\Gamma_u : C\big([t_0,T];\mathbb{R}^n\big) \to C\big([t_0,T];\mathbb{R}^n\big)$ defined by

$$\Gamma_u(x)(t) = (H^{-1}\hat{P}\,u)(t) + H^{-1}TQ_\sigma G^{-1}F\big(\cdot,x(\cdot)\big)(t)$$

for $T \geq t \geq t_0$. It is easy to see that

$$\sup_{t_0 \leq t \leq T}\|\Gamma_u(x)(t) - \Gamma_u(x')(t)\| \leq \overline{\gamma}_T \left\|H^{-1}\right\| \sup_{t_0 \leq t \leq T}\|x(t) - x'(t)\|,$$

for any $x, x' \in C\big([t_0,T];\mathbb{R}^n\big)$. Since $\overline{\gamma}_T \|H^{-1}\| < 1$, Γ_u is a contractive mapping. Hence, by the fixed point theorem, there exists uniquely an $x^* \in C\big([t_0,T];\mathbb{R}^n\big)$ such that

$$x^* = \Gamma_u(x^*).$$

Denote $x^* = g(u)$ we have

$$g(u)(t) = (H^{-1}\hat{P}\,u)(t) + H^{-1}TQ_\sigma G^{-1}F\big(\cdot,g(u(\cdot))\big)(t).$$

Further,

$$\sup_{[t_0,T]}\|g(u)(t) - g(u')(t)\| \leq \beta_T \sup_{[t_0,T]}\|u(t) - u'(t)\| + \overline{\gamma}_T \left\|H^{-1}\right\| \sup_{[t_0,T]}\|g(u)(t) - g(u')(t)\|,$$

with $\beta_T = \|H^{-1}\hat{P}\,\|$. Letting $L_T = \frac{\beta_T}{1-\overline{\gamma}_T\|H^{-1}\|}$ deduces

$$\sup_{t_0 \leq t \leq T}\|g(u)(t) - g(u')(t)\| \leq L_T \sup_{t_0 \leq t \leq T}\|u(t) - u'(t)\|.$$

This means that g is Lipschitz continuous with the Lipschitz coefficient L_T. In particular,

$$\sup_{t_0 \leq t \leq T}\|g(u)(t)\| \leq L_T \sup_{t_0 \leq t \leq T}\|u(t)\|. \tag{4.4}$$

Substituting $x = g(u)$ into (4.3) obtains

$$u^{\Delta}(t) = \big(P^{\Delta} + P_\sigma G^{-1}\overline{B}\big)u(t) + P_\sigma G^{-1}F\big(t, g(u)\big)(t) + P_\sigma G^{-1}\int_{t_0}^{t} K(t,s)g(u)(s)\Delta s. \quad (4.5)$$

Note that for any $T \geq t \geq t_0$, the function $P_\sigma G^{-1}F\big(t, g(u)(t)\big)$ is Lipschitz in u. By applying again Lemma 3.3, we can solve $u(\cdot)$ from (4.5) with the initial condition $u(t_0) = P(t_0)x_0$. Then the solution of (4.1) is given by

$$x(t) = g(u)(t),\ T \geq t \geq t_0. \quad (4.6)$$

Further, by Lemma 3.3

$$\|u(t)\| \leq c\,\|u(t_0)\|\,, \quad T \geq t \geq t_0.$$

Combining (4.4) and (4.6) obtains

$$\|x(t)\| \leq M_T\,\|P(t_0)x(0)\|\,, \quad T \geq t \geq t_0,$$

where $M_T = cL_T$. The proof is complete. □

From Lemma 4.2, it follows that the solution $x(\cdot)$ of the equation (4.1) with the initial condition $P(t_0)(x(t_0)-x_0) = 0$ exists on $[t_0, \infty)$ if $\overline{\gamma}_T\,\|H^{-1}\| < 1$ for all $T > t_0$ and the constant-variation formulas (3.16) can be written as

$$x(t) = \Phi(t,t_0)P(t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)P_\sigma G^{-1}(\tau)\Big(F(\tau,x) + \int_{t_0}^{\tau}K(\tau,s)H^{-1}TQ_\sigma G^{-1}F(\cdot,x)(s)\Delta s\Big)\Delta\tau + H^{-1}TQ_\sigma G^{-1}F(t,x(t)),\quad t \geq t_0. \quad (4.7)$$

To proceed, firstly, we consider the boundedness of solutions of the equation (3.15) under small nonlinear perturbations.

Theorem 4.3. *Assume that the assumptions 3.6, 4.1 hold, the solutions of* (3.15) *is uniformly stable and* H^{-1} *is a bounded operator acting on* $C_{TQ_\sigma G^{-1}}([0,\infty),\mathbb{R}^n)$ *with* $\|H^{-1}\| = K_1$*. Then, if* $L = 1 - K_1\overline{\gamma}_\infty > 0$*, we can find a constant* $M_2 > 0$ *such that the solution* $x(\cdot)$ *of* (4.1) *with the initial condition* (3.14) *satisfies*

$$\|x(t)\| \leq M_2 e^{M_2 N(t)}\|P(t_0)x_0\|, \quad (4.8)$$

for all $t \geq t_0$*, where*

$$N(t) = \int_{t_0}^{t}\Big(l_\tau + K_1\int_{t_0}^{\tau}\overline{\gamma}_s\|P_\sigma G^{-1}K(\tau,s)Q(s)\|\Delta s\Big)\Delta\tau.$$

Proof. Firstly, we note that the condition $L > 0$ implies that the solution $x(\cdot)$ of (4.1) with the initial condition (3.14) exists on $[t_0, \infty)$ by Lemma 4.2. The uniform stability of solutions of (3.15) says that

$$\|\Phi(t,s)\| \leq M_0, \quad t \geq s \geq t_0.$$

Therefore, from the variation of constants formula (4.7), it follows that for all $t \geq t_0$

$$\begin{aligned}\|x(t)\| &\leq M_0\,\|P(t_0)x_0\| + \big\|H^{-1}TQ_\sigma G^{-1}F\big(\cdot,x(\cdot)\big)(t)\big\| \\ &+ \int_{t_0}^{t}M_0\Big(\big\|P_\sigma G^{-1}(\tau)F(\tau,x(\tau))\big\|\int_{t_0}^{\tau}\|P_\sigma G^{-1}(\tau) \\ &+ K(\tau,s)H^{-1}TQ_\sigma G^{-1}(s)F(\cdot,x(\cdot))(s)\|\Delta s\Big)\Delta\tau.\end{aligned}$$

By virtue of the Lipschitz conditions of $P_\sigma G^{-1}F(\cdot,x(\cdot))$ and $Q_\sigma G^{-1}F(\cdot,x(\cdot))$, we get

$$\begin{aligned}\|x(t)\| &\leq M_0\|P(t_0)x_0\| + \big\|H^{-1}\big\|\,\overline{\gamma}_t\sup_{t_0\leq s\leq t}\|x(s)\| \\ &+ M_0\int_{t_0}^{t}\Big(l_\tau\,\|x(\tau)\| + \int_{t_0}^{\tau}\|P_\sigma G^{-1}K(\tau,s)Q(s)\| \\ &\times \|H^{-1}\|\,\overline{\gamma}_s\sup_{t_0\leq s_1\leq s}\|x(s_1)\|\Delta s\Big)\Delta\tau \\ &\leq M_0\,\|P(t_0)x_0\| + K_1\overline{\gamma}_\infty\sup_{t_0\leq s\leq t}\|x(s)\| \\ &+ M_0\int_{t_0}^{t}\Big(l_\tau\sup_{t_0\leq s\leq\tau}\|x(s)\| + K_1\int_{t_0}^{\tau}\overline{\gamma}_s \\ &\times \|P_\sigma G^{-1}K(\tau,s)Q(s)\|\sup_{t_0\leq s_1\leq s}\|x(s)\|\Delta s\Big)\Delta\tau.\end{aligned}$$

Putting $M_2 = \frac{M_0}{L}$, we have

$$\sup_{t_0\le s\le t}\|x(s)\| \le M_2\,\|P(t_0)x_0\| + M_2\int_{t_0}^{t}\Big(l_\tau \sup_{t_0\le s\le\tau}\|x(s)\| + K_1\int_{t_0}^{\tau}\overline{\gamma}_s\times \|P_\sigma G^{-1}(\tau)K(\tau,s)Q(s)\| \sup_{t_0\le s_1\le s}\|x(s_1)\|\Delta s\Big)\Delta\tau.$$

Following the generalized Gronwall-Bellman inequality in Lemma 2.6

$$\|x(t)\|\le \sup_{t_0\le s\le t}\|x(s)\|\le M_2\|P(t_0)x_0\|e_{N_1(\cdot)}(t,t_0)$$

for all $t \ge t_0$, where

$$N_1(\tau)=l_\tau+\int_{t_0}^{\tau}K_1\overline{\gamma}_s\|P_\sigma G^{-1}(\tau)K(\tau,s)Q(s)\|\Delta s.$$

Since $N_1(\cdot)$ is positive,

$$e_{N_1(\cdot)}(t,t_0) \le \exp\Big(\int_{t_0}^{t} N_1(\tau)\Delta\tau\Big) \le \exp\Big(\int_{t_0}^{t}\big(l_\tau + K_1\int_{t_0}^{\tau}\overline{\gamma}_s\|P_\sigma G^{-1}(\tau) \times K(\tau,s)Q(s)\|\Delta s\big)\Delta\tau\Big).$$

Thus, $\|x(t)\| \le M_2 e^{N(t)}\|P(t_0)x_0\|$ for all $t \ge t_0$. The proof is complete. □

As a consequence of Theorem 4.3 we see that

Corollary 4.4. *Assume that the assumptions 3.6, 4.1 hold, the solutions of* (3.15) *is uniformly stable and* H^{-1} *is a bounded operator acting on* $C_{TQ_\sigma G^{-1}}([0,\infty),\mathbb{R}^n)$ *with* $\|H^{-1}\| = K_1$. *If* $L = 1 - K_1\overline{\gamma}_\infty > 0$ *and*

$$N=\int_{t_0}^{\infty}\Big(l_\tau+\int_{t_0}^{\tau}K_1\overline{\gamma}_s\|P_\sigma G^{-1}K(\tau,s)Q(s)\|\Delta s\Big)\Delta\tau < \infty,$$

then, the solution of the equation (4.1) *is uniformly stable in the sense that*

$$\|x(t)\| \le M_3\|P(t_0)x_0\|,\quad t\ge t_0,$$

for a certain constant M_3.

Next, we consider the robust exponential stability of (3.15) For any $\lambda > 0$, let $\widetilde{G} = G(I + \lambda\mu Q_\sigma)$; $\widetilde{K}(t,h) = e_\lambda(\sigma(t),s)K(t,h)e_{\ominus\lambda}(h,s)$. We define the operators $\widetilde{H}$ as in (3.6) by using $\widetilde{G}$ and $\widetilde{K}(t,h)$ instead of G and $K(t,h)$. Then we have the following theorem about exponential stability of solutions of the equation (3.15) under small nonlinear perturbations.

Theorem 4.5. *If the equation* (3.15) *is* ω-*exponentially stable and there exists an* $\lambda \in (0,\omega)$ *and* $\ominus\lambda \in \mathcal{R}^+$ *such that* $\widetilde{H}^{-1}$ *acts continuously on* $C_{TQ_\sigma G^{-1}}([0,\infty),\mathbb{R}^n)$ *with* $\|\widetilde{H}^{-1}\| = \widetilde{K}_1$ *satisfying* $\widetilde{L} = 1 - \widetilde{K}_1\overline{\gamma}_\infty > 0$. *Suppose further that*

$$\limsup_{\tau\to\infty}\Big(l_\tau + \widetilde{K}_1\int_{t_0}^{\tau}\overline{\gamma}_h e_\lambda(\tau,h)\|P_\sigma G^{-1}(\tau) \times K(\tau,h)Q(h)\|\Delta h\Big) \le \frac{\lambda\widetilde{L}}{2M(1+\lambda\mu^*)} \quad (4.9)$$

with M *defined in* (3.22). *That is, there is a positive number* ω_1 *such that the perturbed equation* (4.1) *is* ω_1-*exponentially stable.*

Proof. Let ε_0 be a positive number such that $\delta + \varepsilon_0 \le \frac{\lambda}{2(1+\lambda\mu^*)}$. Then, from (4.9), there is a positive number $T_0 > 0$ such that

$$l_t + \widetilde{K}_1\int_{t_0}^{t}\overline{\gamma}_h e_\lambda(t,h)\|P_\sigma G^{-1}(t)K(t,h)Q(h)\|\Delta h < \delta + \varepsilon_0 \le \frac{\lambda}{2(1+\lambda\mu^*)}, \quad (4.10)$$

for all $t \ge T_0$. To simplify notations we write $x(t)$ for the solution $x(t,s,x_0)$.

$$A_\sigma(t)(Px)^\Delta(t) = \overline{B}(t)x(t) + \int_{t_0}^{t}K(t,s)x(s)\Delta s + F(t,x(t)).$$

Let $y(t) = e_\lambda(t,s)x(t), t\ge s\ge t_0$. Since

$$y^\Delta(t) = e_\lambda(\sigma(t),s)x^\Delta(t) + \lambda e_\lambda(t,s)x(t)$$

It is easy to see that y satisfies the equation

$$\begin{aligned}
&A_\sigma(t)(Py)^\Delta(t) = A_\sigma(t)(e_\lambda(t,s)Px)^\Delta(t)\\
&= A_\sigma(t)\Big(e_\lambda(\sigma(t),s)(Px)^\Delta + \lambda e_\lambda(t,s)Px\Big)\\
&= e_\lambda(\sigma(t),s)\Big(\overline{B}(t)x(t) + \int_{t_0}^t K(t,s)x(s)\Delta s\\
&+ F(t,x(t))\Big) + \lambda e_\lambda(t,s)A_\sigma(t)Px\\
&= \big[(1+\lambda\mu(t))\overline{B}(t) + \lambda A_\sigma(t)P(t)\big]y(t)\\
&+ \int_s^t e_\lambda(\sigma(t),s)K(t,h)e_{\ominus\lambda}(h,s)y(h)\Delta h\\
&+ e_\lambda(\sigma(t),s)F\big(t, e_{\ominus\lambda}(t,s)y(t)\big)\\
&= \overline{\overline{B}}y(t) + \int_s^t \widetilde{K}(t,h)y(h)\Delta h + \widetilde{F}\big(t,y(t)\big),
\end{aligned}$$

for all $t \geq s$ and (4.11)

$$\begin{aligned}
\overline{\overline{B}}(t) &= (1+\lambda\mu(t))\overline{B}(t) + \lambda A_\sigma(t)P(t),\\
\widetilde{K}(t,h) &= e_\lambda(\sigma(t),s)K(t,h)e_{\ominus\lambda}(h,s),\\
\widetilde{F}\big(t,y(t)\big) &= e_\lambda(\sigma(t),s)F\big(t, e_{\ominus\lambda}(t,s)y(t)\big).
\end{aligned}$$

Since

$$\begin{aligned}
\widetilde{G} &= A_\sigma - \Big[(1+\lambda\mu)\overline{B} + \lambda A_\sigma P\Big]TQ_\sigma\\
&= G - \lambda\mu BTQ_\sigma = G\big(I - \lambda\mu G^{-1}BTQ_\sigma\big)\\
&= G(I + \lambda\mu Q_\sigma).
\end{aligned}$$

We see that $(I+\lambda\mu Q_\sigma)^{-1} = \big(P_\sigma + (1+\lambda\mu)Q_\sigma\big)^{-1} = P_\sigma + \frac{1}{1+\lambda\mu}Q_\sigma$, which implies $\widetilde{G}$ is invertible, and $\widetilde{G}^{-1} = \big(P_\sigma + \frac{1}{1+\lambda\mu}Q_\sigma\big)G^{-1}$, it is seen that the equation (4.11) is index-1. Furthermore,

$$P_\sigma\widetilde{G}^{-1} = P_\sigma G^{-1};\ \ (1+\lambda\mu)Q_\sigma\widetilde{G}^{-1} = Q_\sigma G^{-1}.$$

Moreover,

$$\begin{aligned}
P_\sigma\widetilde{G}^{-1}\widetilde{F}(t,y(t)) &= e_\lambda(\sigma(t),s)\\
&\quad\times P_\sigma G^{-1}F\big(t, e_{\ominus\lambda}(t,s)y(t)\big),\\
Q_\sigma\widetilde{G}^{-1}\widetilde{F}(t,\cdot) &= e_\lambda(t,s)Q_\sigma G^{-1}F\big(t, e_{\ominus\lambda}(t,s)y(t)\big).
\end{aligned}$$

Further, $P_\sigma\widetilde{G}^{-1}\widetilde{F}(t,\cdot)$ and $TQ_\sigma\widetilde{G}^{-1}\widetilde{F}(t,\cdot)$ are $(1+\lambda\mu(t))l_t$ and γ_t-Lipschitz, respectively. Consider the corresponding homogeneous equation to (4.11)

$$A_\sigma(t)(Pz)^\Delta(t) = \overline{\overline{B}}(t)z(t) + \int_s^t \widetilde{K}(t,h)z(h)\Delta h. \tag{4.12}$$

By definition, the Cauchy operator $\widetilde{\Phi}(t,h), t \geq h \geq s$ of (4.12) and $\Phi(t,h)$ of (3.15) have a relation

$$\widetilde{\Phi}(t,h) = e_\lambda(t,h)\Phi(t,h),\ \ t \geq h \geq s.$$

Therefore, for all $t \geq h \geq s$

$$\|\widetilde{\Phi}(t,h)\| = e_\lambda(t,h)\,\|\Phi(t,h)\| \leq Me_{(\lambda\ominus\omega)}(t,h) \leq M.$$

This means that (4.12) is uniformly stable. And the solution of (4.11) is expressed by

$$\begin{aligned}
y(t) = \widetilde{\Phi}(t,s)P(s)y_0 + \int_s^t &\widetilde{\Phi}(t,\tau)P_\sigma\widetilde{G}^{-1}(\tau)\Big(\widetilde{F}(\tau,y(\tau))\\
&+ \int_s^\tau \widetilde{K}(\tau,h)\widetilde{H}^{-1}TQ_\sigma\widetilde{G}^{-1}\widetilde{F}(\cdot,x(\cdot))(h)\Delta h\Big)\Delta\tau\\
&+ \widetilde{H}^{-1}TQ_\sigma\widetilde{G}^{-1}\widetilde{F}(t,y(t)).
\end{aligned}$$

$$\begin{aligned}
\|y(t)\| \leq\, &MP(s)y_0 + \widetilde{K}_1\overline{\gamma}_\infty \sup_{t_0\leq h\leq t}\|y(h)\|\\
&+M\int_s^t \Big((1+\lambda\mu(\tau))l_\tau\|y(\tau)\|\\
&+\int_s^\tau \|P_\sigma\widetilde{G}^{-1}(\tau)\widetilde{K}(\tau,h)Q(h)\|\|\\
&\times \widetilde{K}_1\|\overline{\gamma}_h \sup_{t_0\leq h_1\leq h}\|x(h_1)\|\Delta h\Big)\Delta\tau.
\end{aligned}$$

By using Theorem 4.3, with $\widetilde{M} = \frac{M}{L}$ we have

$$\|y(t)\| \leq \widetilde{M}e_{\widetilde{N}(\cdot,s)}(t,s)\,\|P(s)x_0\|,\ t \geq s.$$

where

$$\begin{aligned}
\widetilde{N}(\tau,s) = (1+\lambda\mu(\tau))\Big(l_\tau + \widetilde{K}_1\int_s^\tau \overline{\gamma}_h e_\lambda(\tau,h)\\
\times\|P_\sigma G^{-1}(\tau)K(\tau,h)Q(h)\|\Delta h\Big)
\end{aligned}$$

• Consider three cases when $t \geq T_0 \geq s \geq t_0$. From (4.10) we see that

$$\begin{aligned} \|x(t)\| &= e_{\ominus\lambda}(t,s)\,\|y(t)\| \\ &\leq M e_{\ominus\lambda}(t,s) e_{\widetilde{N}}(t,s)\,\|P(s)x_0\| \\ &= M e_{\ominus\lambda}(t,s) e_{\widetilde{N}}(t,T_0) e_{\widetilde{N}}(T_0,s)\,\|P(s)x_0\| \\ &\leq M e_{\ominus\lambda}(t,s) e_{\widetilde{N}}(t,s) e_{\widetilde{N}}(T_0,t_0)\,\|P(s)x_0\| \\ &= M e_{\widetilde{N}\ominus\lambda}(t,s) e_{\widetilde{N}}(T_0,t_0)\,\|P(s)x_0\| \end{aligned}$$

where

$$\begin{aligned} \widetilde{N} \ominus \lambda &= (1+\lambda\mu(\tau))\Big[l_\tau + \widetilde{K}_1 \int_s^\tau \overline{\gamma}_h e_\lambda(\tau,h)\| \\ &\quad \times P_\sigma G^{-1}(\tau) K(\tau,h) Q(h)\|\Delta h\Big] \ominus \lambda \\ &\leq (1+\lambda\mu(\tau))(\delta+\varepsilon_0) \ominus \lambda \\ &= \frac{(\delta+\varepsilon_0)[1+\mu(\tau)\lambda] - \lambda}{1+\mu(\tau)\lambda} \\ &\leq \frac{-\lambda/2}{1+\mu(\tau)\lambda} \leq \frac{-\lambda/2}{1+\mu^*\lambda} := -\lambda_1. \end{aligned}$$

Thus, $\|x(t)\| \leq K_1 e_{-\lambda_1}(t,s)\|P(s)x_0\|$, where $K_1 = M\,\mathrm{e}_{\widetilde{N}(\cdot)}(T_0,t_0)$.

• In case $t > s > T \geq t_0$, using a similar argument as above we get

$$\|x(t)\| \leq M\|x(s)\| e_{-\lambda_1}(t,s).$$

• Consider the remaining case $t_0 \leq s \leq t \leq T_0$. With $\lambda_1 > 0$ defined above and from the inequality (4.5), we have By virtue of the positivity of $\widetilde{L}$ and Lemma 4.2 we get

$$\|x(t)\| \leq M_{T_0} e_{\frac{\lambda}{2}}(t,s) e_{\ominus\frac{\lambda}{2}}(t,s)\|x(s)\|.$$

Since $\ominus\frac{\lambda}{2} = \frac{-\lambda/2}{1+\mu(\tau)\frac{\lambda}{2}} \leq \frac{-\lambda/2}{1+\mu^*\lambda} := -\lambda_1$. Thus

$$\|x(t)\| \leq M_{T_0} e_{\frac{\lambda}{2}}(T_0,t_0) e_{-\lambda_1}(t,s)\|x(s)\|.$$

Combining the above estimates yields

$\|x(t)\| \leq K e_{\ominus\omega_1}(t,s)\|x(s)\|$ for all $t \geq s \geq t_0$,

where $K = \max\{M, K_1, M_{T_0} e_{\frac{\lambda}{2}}(T_0,t_0)\}$. The proof is complete. □

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