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## TAP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO<br>
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NONLOCAL RAYLEIGH-STOKES FLOWS: EXISTENCE AND<br>
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LEIGH-STOKES FLOWS: EXISTENCE ANI<br>
REGULARITY RESULTS<br>
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### Keywords:



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mann–Liouville. giải t<br>
Từ khóa:<br>  $S\psi$  tồn tại nghiệm, Tính chính<br>
quy, Nghiệm tích phân, Bài toán<br>
Rayleigh-Stokes, Dạo hàm Rie-<br>
mann-Liouville.<br>
1 INTRODUCTION<br>
The article studies the following general Cauchy Time khoise:<br>  $S\psi$  tồn tại nghiệm, Tính chính<br>
quy, Nghiệm tích phân, Bài toán<br>
Rayleigh-Stokes, Dạo hàm Rie-<br>
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1 INTRODUCTION ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the fractional derivative of order<br>
The a

problem

*Rayleigh–Stokes, Dao hàm Rie*  
\n
$$
\begin{array}{ll}\n\text{may} & \text{and } \text{any, } T \in (0, \infty), \text{ and } \partial \text{ fractional derivative of } \\ \text{The article studies the following general Cauchy (Kilbas, 2006; Y. Zhou, problem)} \\
\downarrow \partial_t u = -(1 + k\partial_t^{\alpha}) Au + g(t), t > 0 \\
u(0) = u_0\n\end{array}\n\qquad\n\begin{array}{ll}\n\partial_t^{\alpha} u(t, x) \\
u(0) = u_0\n\end{array}\n\qquad\n\begin{array}{ll}\n\partial_t u = -(1 + k\partial_t^{\alpha}) Au + g(t), t > 0 \\
u(0) = u_0\n\end{array}\n\qquad\n\begin{array}{ll}\n\partial_t^{\alpha} u(t, x) \\
\downarrow \text{or the fol-} \\
\text{using general second-order fluid model:} \\
(0, T). Here, \Gamma(\cdot) \text{ is the } \\ \n\begin{array}{ll}\n\partial_t u - (1 + k\partial_t^{\alpha}) \Delta u = f(t, u), \quad x \in \Omega, \\
u(x) = 0 \\
\end{array}\n\qquad\n\begin{array}{ll}\n\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \quad I \\
D(A) = H^2(\Omega) \cap H_0^1(\Omega) \end{array}
$$

fractional derivative of a problem		
The article studies the following general Cauchy problem	(Kilbas, 2006; Y. Zhou problem	
$\begin{cases}\n\partial_t u = -(1 + k\partial_t^{\alpha}) Au + g(t), t > 0 \\ u(0) = u_0\n\end{cases}$ \n	(1.1)	$= \frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1 - \alpha)} \right)$ \n
arises from the Rayleigh-Stokes problem for the fol- lowing general second-order fluid model: (0, T). Here, $\Gamma(\cdot)$ is the function (0, T). Here, $\Gamma(\cdot)$ is the function (1.1), which where $\Gamma(2) = \int_0^\infty t^{z-1} e^{-t} dt$ .		
$\begin{cases}\n\partial_t u - (1 + k\partial_t^{\alpha}) \Delta u = f(t, u), & x \in \Omega, \\ u(t, x) = 0, & x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega\n\end{cases}$ \n	$E(\alpha) = \frac{D(A)}{\alpha} = \frac{H^2(\Omega) \cap H_0^1}{H^2(\Omega)} \text{ and } f : [0, T] \times H^2(\Omega)$ \n	
$u(t, x) = 0, \quad x \in \Omega$ \n	$x \in \Omega$ \n	$L^2(\Omega)$ and $f : [0, T] \times H^2(\Omega)$
$u(t, x) = u_0(x), \quad x \in \Omega$ \n	$x \in \Omega$ \n	$L^2(\Omega)$ and $f : [0, T] \times H^2(\Omega)$
$u(t, x) = 0, \quad x \in \Omega$ \n	$L^2(\Omega)$ and $f : [0, T] \times H^2(\Omega)$	

ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the Riemann-Liouville<br>fractional derivative of order  $\alpha \in (0, 1)$  defined by<br>(Kilbas, 2006; Y. Zhou, 2014): ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the Riemann-Liouville is the Riemann-Liouville<br>der  $\alpha \in (0,1)$  defined by<br>014): ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the Riemann-Liouville<br>fractional derivative of order  $\alpha \in (0, 1)$  defined by<br>(Kilbas, 2006; Y. Zhou, 2014): ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the Riemann-Liouville<br>fractional derivative of order  $\alpha \in (0, 1)$  defined by<br>(Kilbas, 2006; Y. Zhou, 2014):<br> $\partial_t^{\alpha}u(t, x)$ 

(1.1) arises from the Rayleigh-Stokes problem for the fol-∂<sup>α</sup> <sup>t</sup> <sup>u</sup>(t, x) <sup>=</sup> ∂ ∂t 1 Γ(1 <sup>−</sup> <sup>α</sup>) <sup>t</sup> <sup>0</sup> (<sup>t</sup> <sup>−</sup> <sup>s</sup>)−<sup>α</sup> u(s, x)ds (1.3)

(1.2)  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ . Then, we rewrit  $\label{eq:2.1} \left\{ \begin{aligned} \partial_t u &= -\left(1 + k \partial_t^\alpha\right) A u + g(t), t > 0 &\quad \text{(1.1)}\\ u(0) &= u_0 &\quad \text{(1.1)}\\ \end{aligned} \right.\\ \begin{aligned} &\text{arises from the Rayleigh-Stokes problem for the fol-}\\ \text{using general second-order fluid model:}\\ &\text{(0, $T$), Here, $\Gamma(\cdot)$ is the}\\ \partial_t u - \left(1 + k \partial_t^\alpha\right) \Delta u &= f(t,u), &\quad x \in \Omega, &\quad \Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt. \text{ L}\\ u(t,x) &= 0, &\quad$ tokes problem for the fol-<br>  $f(t, u)$ ,  $x \in \Omega$ , with the right-hand side definition<br>  $f(t, u)$ ,  $x \in \Omega$ ,  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ . Let  $A = x \in \partial\Omega$ ,  $(1.2)$ ,  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ . Then  $x \in \Omega$  in the form  $(1.1)$ , where  $u_$ ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the Riemann-Liouville<br>fractional derivative of order  $\alpha \in (0, 1)$  defined by<br>(Kilbas, 2006; Y. Zhou, 2014):<br> $\partial_t^{\alpha} u(t, x)$ <br>=  $\frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s, x) ds \right)$  (1.3)<br>with ary,  $T \in (0, \infty)$ , and  $\partial_t^{\alpha}$  is the Riemann-Liouville<br>fractional derivative of order  $\alpha \in (0, 1)$  defined by<br>(Kilbas, 2006; Y. Zhou, 2014):<br> $\partial_t^{\alpha}u(t, x)$ <br>=  $\frac{\partial}{\partial t}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(s, x)ds\right)$  (1.3)<br>with t fractional derivative of order  $\alpha \in (0, 1)$  defined by<br>
(Kilbas, 2006; Y. Zhou, 2014):<br>  $\partial_t^{\alpha} u(t, x)$ <br>  $= \frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s, x) ds \right)$  (1.3)<br>
with the right-hand side defined at each point in<br>
(0, T).  $\int_0^\infty t^{z-1}e^{-t}dt$ . Let  $A = -\Delta$  in  $L^2(\Omega)$  with , and  $\partial_t^{\alpha}$  is the Riemann-Liouville<br>tive of order  $\alpha \in (0, 1)$  defined by<br>T. Zhou, 2014):<br> $\frac{1}{-\alpha} \int_0^t (t-s)^{-\alpha} u(s, x) ds$  (1.3)<br>and side defined at each point in<br>c) is the Gamma function, that is,<br> $e^{-t}dt$ . Let  $A = -\$ (Kilbas, 2006; Y. Zhou, 2014):<br>  $\partial_t^{\alpha} u(t, x)$ <br>
=  $\frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s, x) ds \right)$  (<br>
with the right-hand side defined at each poin<br>
(0, T). Here,  $\Gamma(\cdot)$  is the Gamma function, tha<br>  $\Gamma(z) = \int_0^{\infty} t^{z-1}$ f order  $\alpha \in (0,1)$  defined by<br>
1, 2014):<br>  $\int_0^t (t-s)^{-\alpha} u(s, x) ds$  (1.3)<br>
ide defined at each point in<br>
ne Gamma function, that is,<br>
Let  $A = -\Delta$  in  $L^2(\Omega)$  with<br>
( $\Omega$ ). Then, we rewrite (1.2)<br>
ere  $u_0$  is the initial va (Kilbas, 2006; Y. Zhou, 2014):<br>  $\partial_t^{\alpha} u(t, x)$ <br>  $= \frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s, x) ds \right)$  (1.3)<br>
with the right-hand side defined at each point in<br>  $(0, T)$ . Here,  $\Gamma(\cdot)$  is the Gamma function, that is,<br>  $\Gamma(z) = \int$  $L^2(\Omega)$  and  $f: [0,T] \times L^2(\Omega) \to L^2(\Omega)$  is a given  $\int_{0}^{\alpha} u(t, x)$ <br>=  $\frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-s)^{-\alpha} u(s, x) ds \right)$  (1.3)<br>th the right-hand side defined at each point in<br>T). Here,  $\Gamma(\cdot)$  is the Gamma function, that is,<br> $z$ ) =  $\int_{0}^{\infty} t^{z-1} e^{-t} dt$ . Let  $A = -\Delta$  in (1.3)<br>
(1.3)<br>
(a) each point in<br>
(a) intinual is,<br>
in  $L^2(\Omega)$  with<br>
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intial value in<br>
(0) is a given<br>
ons.  $\begin{aligned}\n\sigma_t u(t,x) &= \frac{\partial}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s,x) ds \right) (1.3) \\
\text{with the right-hand side defined at each point in } (0,T). \text{ Here, } \Gamma(\cdot) \text{ is the Gamma function, that is, } \Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt. \text{ Let } A = -\Delta \text{ in } L^2(\Omega) \text{ with } D(A) &= H^2(\Omega) \cap H_0^1(\Omega). \text{ Then, we rewrite (1.2)} \\
\text{in the form (1.1), where } u_0 \text{ is the initial value in } L^2(\Omega$ =  $\frac{0}{\partial t} \left( \frac{1}{\Gamma(1-\alpha)} \int_0^{\alpha} (t-s)^{-\alpha} u(s, x) ds \right)$  (1.3)<br>with the right-hand side defined at each point in<br>(0, T). Here,  $\Gamma(\cdot)$  is the Gamma function, that is,<br> $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ . Let  $A = -\Delta$  in  $L^2(\Omega)$  with

*Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma > \text{Maxwell fluids}, \text{ and second-order fluids, has at-} \text{ contour and \text{tracted significant attention due to their physi-} \times \mathcal{K}_r(r) = \frac{k}{r} \frac{\lambda_n r^{\alpha} \sin \theta}{r^{\alpha} r^{\beta}}$ *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$ <br>Maxwell fluids, and second-order fluids, has at-<br>contour and<br>tracted significant attenti *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma >$ <br>Maxwell fluids, and second-order fluids, has atcontour and<br>tracted significant attention due *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$ <br>Maxwell fluids, and second-order fluids, has at-<br>contour and<br>tracted significant attenti *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$ <br>Maxwell fluids, and second-order fluids, has atcontour and<br>tracted significant attention Nguyen Nhu Quan/Vol 10. No 4\_August 2024| p.26-3:<br>
non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma,$ <br>
Maxwell fluids, and second-order fluids, has atcontour and<br>
tracted significant attention due t *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-3<br>non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma$ ,<br>Maxwell fluids, and second-order fluids, has atcontour and<br>tracted significant attention due to *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>
non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma >$ <br>
Maxwell fluids, and second-order fluids, has atcontour and<br>
tracted significant attention non-Newtonian fluids, such as Oldroyd-B fluids, where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$ <br>Maxwell fluids, and second-order fluids, has atcontour and<br>tracted significant attention due to their physi-<br>cal importance (Fetecau, 2003). ids. caused by a sudden displacement of the boundary.<br>
The mathematical model is derived by combining<br>
the laws of conservation in physics with the consti-<br>
tutive relationship of second-order viscoelastic flu-<br>
Lemma 2.1. The The mathematical model is derived by combining  $r > 0$ , and  $\lambda_n, n > 0$ , we have  $\lambda$ <br>the laws of conservation in physics with the consti-<br>tutive relationship of second-order viscoelastic flu-<br>is  $\rho\left(\frac{d\mathbf{V}}{dt} - (\mathbf{V} \cdot$ 

$$
\rho \left( \frac{d\mathbf{V}}{dt} - (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = \nabla \cdot (-p\mathbf{I} + \mu_0 \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2)
$$

the<br>matical model is derived by combining<br>
of conservation in physics with the consti-<br>
lationship of second-order viscoelastic flu-<br>
have<br>
have<br>  $\frac{dV}{dt} - (\mathbf{V} \cdot \nabla) \mathbf{V} = \nabla \cdot (-p\mathbf{I} + \mu_0 \mathbf{A}_1$ <br>  $(i) \delta$ <br>  $+ \alpha_1 \math$ is derived by combining<br>
n physics with the consti-<br>
nd-order viscoelastic flu-<br>
Lemma 2.1. T.<br>
have the followir<br>  $\begin{aligned}\n(i) S_n(\alpha, 0) &= 1; \\
(ii) There exists \\
+\alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2\end{aligned}$ that<br>
he fluid, **V** is the velocity,<br>
ure,  $\mu_$ tutive relationship of second-order viscoelastic f<br>
ids.<br>  $\rho \left( \frac{d\mathbf{V}}{dt} - (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = \nabla \cdot (-p\mathbf{I} + \mu_0 \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2)$ <br>
where  $\rho$  is the density of the fluid, **V** is the veloci<br>  $p$  i  $\frac{d\mathbf{V}}{dt} - (\mathbf{V} \cdot \nabla) \mathbf{V}$  =  $\nabla \cdot (-p\mathbf{I} + \mu_0 \mathbf{A}_1$  (i)  $S_n$ <br>  $+ \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2$  that<br>  $\nu$  is the density of the fluid, **V** is the velocity,<br>  $\therefore$  hydrostatic pressure,  $\mu_0 \ge 0$ ,  $\alpha_1 \ge 0$ , an (*u)* There exists a<br>
where  $\rho$  is the density of the fluid, **V** is the velocity,<br>  $p$  is the hydrostatic pressure,  $\mu_0 \ge 0$ ,  $\alpha_1 \ge 0$ , and  $\mu_0 \ge 0$ ,  $P_{\text{root}}$ . Property (i)<br>  $\alpha_1 + \alpha_2 = 0$ . **A**<sub>1</sub> and **A**<sub>2</sub> are t

$$
\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T,
$$
  

$$
\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \mathbf{A}_1
$$

p is the hydrostatic pressure,  $\mu_0 \ge 0$ ,  $\alpha_1 \ge 0$ , and  $\mu_0 \ge 0$ . Property (i) has been pro<br>  $\alpha_1 + \alpha_2 = 0$ . A<sub>1</sub> and A<sub>2</sub> are the kinematic tensors 2015, Theorem 2.2), and Prop<br>
defined by:<br>  $\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla$  $\alpha_1 + \alpha_2 = 0$ . **A**<sub>1</sub> and **A**<sub>2</sub> are the kinematic tensors 2015, Theorem 2.2), and F<br>defined by:<br>  $\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T$ , From Lemma 2.1(ii), it can<br>
bounded in  $L^2(\Omega)$  for all  $t \ge$ <br>  $\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} +$ defined by:<br>  $\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T$ , From Lemma 2.1(ii), it can<br>
bounded in  $L^2(\Omega)$  for all  $t \ge$ <br>  $\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \mathbf{A}_1$ .<br>
Lemma 2.2. Let  $\{\mathcal{P}_{\alpha}(t)\}_{t \ge}$ <br>
perators  $\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T$ ,<br>  $\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla \mathbf{V}) + (\nabla \mathbf{V})^T \mathbf{A}_1$ .<br>
bounded in  $L^2(\Omega)$  for all t<br>
bounded in  $L^2(\Omega)$  for all t<br>
bounded in  $L^2(\Omega)$  for all t<br>
bounded in  $L^2(\Omega)$  for all dt<br>
MINARIES<br>
oper<br>
MINARIES<br>
(*i*)<br>
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note  $B(X)$  as the space of all bounded<br>  $[\delta, \infty]$ <br>
rators on X. The special beta function<br>  $\times (0, \infty) \rightarrow (0, \infty)$  is defined by:<br>  $\mathcal{B}(a, b) = \int_0^1 (1 - s)^{a-1} s^{b-1} ds$ 2 PRELIMINARIES<br>
We introduce some notation. For any Banacl<br>
X, we denote  $B(X)$  as the space of all bo<br>
linear operators on X. The special beta fu<br>  $B:(0,\infty)\times(0,\infty)\to(0,\infty)$  is defined by:<br>  $\mathcal{B}(a,b) = \int_0^1 (1-s)^{a-1} s^{b-1} ds$ by the space of all bounded  $B(X)$  as the space of all bounded  $[\delta, \infty)$ ;<br>
s on X. The special beta function  $(ii)$  For  $\epsilon$ <br>  $(\infty) \rightarrow (0, \infty)$  is defined by:<br>  $\mathcal{P}_{\epsilon}$ <br>  $\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ .<br>
Lemma<br>  $\mathcal{B}(a, b) = \frac{\Gamma$ 

$$
\mathcal{B}(a,b) = \int_0^1 (1-s)^{a-1} s^{b-1} ds
$$
 where

$$
\mathcal{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
$$

X, we denote  $B(X)$  as the space of all bounded  $[\delta, \infty)$ ;<br>
linear operators on X. The special beta function (ii) For every  $v \in D(A)$  and  $t \ge$ <br>  $\mathcal{B} : (0, \infty) \times (0, \infty) \to (0, \infty)$  is defined by:<br>  $\mathcal{P}_{\alpha}(t)v = v - \int_0^t \eta(t-s)$  $\begin{aligned} &\text{linear operators on } X. \text{ The special beta function} &\quad(ii) \textit{ For every } v \in D(A) \textit{ and } t \\ &\mathcal{B}: (0,\infty) \times (0,\infty) \to (0,\infty) \textit{ is defined by:} \\ &\mathcal{B}(a,b)=\int_0^1 (1-s)^{a-1}s^{b-1}ds \qquad &\textit{where } \eta(t)=1+\frac{k}{\Gamma(1-\alpha)}t^{-\alpha}.\\ &\text{We have} &\mathcal{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. &\textbf{Lemma 2.3.} \textit{ For every } \delta>\\ &\mathcal{B}(a,b)=\frac{\Gamma(a$  $\begin{aligned} \mathcal{B}: (0,\infty)\times (0,\infty) \rightarrow (0,\infty) \text{ is defined by:}\\ \mathcal{B}(a,b)&=\int_0^1(1-s)^{a-1}s^{b-1}ds\\ \text{We have}\\ \mathcal{B}(a,b)&=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \end{aligned} \begin{aligned} \text{Lermma 2.3. } For\ every\ \delta>0\\ \text{Lermma 2.3. } For\ every\ \delta>0\\ \mathcal{P}'_\alpha(t)v\ \text{and}\ \mathcal{AP}_\alpha(t)v\ \text{are Lipschitz}\\ \text{In the rest of this section, we first introduce a family\\ \delta\leq t\leq T. \ \text{Moreover,$  $\mathcal{P}_{\alpha}(t)v = v - \int_{0}^{\infty} \eta(t-s)A$ <br>  $\mathcal{B}(a,b) = \int_{0}^{1} (1-s)^{a-1}s^{b-1}ds$ <br>
where  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)}t^{-\alpha}$ .<br>
We have<br>  $\mathcal{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ .<br>
Lemma 2.3. For every  $\delta > 0$ <br>  $\mathcal{P}'_{\alpha}(t)v$  and  $A\mathcal{P}_{\alpha}(t)v$  are Lipschi<br> per. We have<br>  $\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ .<br>  $\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ .<br>
Lemma 2.3. For every  $\delta > \mathcal{P}'_a(t)v$  and  $AP_\alpha(t)v$  are Lipsc<br>
In the rest of this section, we first introduce a fam-<br>  $\delta \le t \le T$ . Moreover, for  $0 < s$ <br>
ily We have  $\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ .<br>
Lemma 2.3. For every  $\delta > 0$  a<br>  $\mathcal{P}'_{\alpha}(t)v$  and  $A\mathcal{P}_{\alpha}(t)v$  are Lipschitz<br>
In the rest of this section, we first introduce a fam-<br>
ily of operators and then use representation f  $p'_{\alpha}(t)v$  and  $AP_{\alpha}(t)$ <br>
ion, we first introduce a fam-<br>
ion, we first introduce a fam-<br>
ion, we first introduce a fam-<br>
ion is family of<br>
defined by<br>  $p'_{\alpha}(t)v$  and  $AP_{\alpha}(t)$ <br>  $\delta \leq t \leq T$ . Mor<br>
defined by<br>  $\|\mathcal{P}'_{\alpha}($ nd then use representation formu-<br>
main estimates for this family of<br>
will be used throughout the pa-<br>
and<br>
ults from (Bazhlekova, 2015;  $||A\mathcal{P}_{\alpha}(t) - A\mathcal{P}_{\alpha}(t)||$ <br>
2021), we introduce a family of<br>  $\Big|_{t\geq 0}$  defined

operators  $\{\mathcal{P}_{\alpha}(t)\}_{t\geq 0}$  defined by which will be used throughout the pa-<br>
results from (Bazhlekova, 2015;<br>
u Y., 2021), we introduce a family of<br>  $P_{\alpha}(t)$ <sub> $t\geq 0$ </sub> defined by<br>  $\begin{aligned}\n\mathcal{P}_{\alpha}(t) &= \sum_{n=1}^{\infty} \mathcal{S}_n(\alpha, t) (v, \varphi_n) \varphi_n, \quad (2.1) \quad \text{Le} \\
\{\mathcal{P}_{$ 

$$
\mathcal{P}_{\alpha}(t)v = \sum_{n=1}^{\infty} \mathcal{S}_n(\alpha, t) (v, \varphi_n) \varphi_n, \qquad (2.1) \quad \text{Lemma}
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\n
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\{\mathcal{P}_{\alpha}(t)\}_t,
$$

with  $v \in L^2(\Omega)$ , where

g the results from (Bazhlekova, 2015;  
\n[1. Zhou Y., 2021), we introduce a family of  
\nators 
$$
\{\mathcal{P}_{\alpha}(t)\}_{t\geq 0}
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 defined by  
\n
$$
\mathcal{P}_{\alpha}(t)v = \sum_{n=1}^{\infty} \mathcal{S}_n(\alpha, t) (v, \varphi_n) \varphi_n,
$$
\n(2.1) Lemma 2.4.  
\n
$$
v \in L^2(\Omega), \text{ where}
$$
\n
$$
\mathcal{S}_n(\alpha, t) = \frac{1}{2\pi i} \int_{Br} e^{zt} \frac{1}{z + k\lambda_n z^{\alpha} + \lambda_n} dz
$$
\n
$$
= \int_0^{\infty} e^{-rt} \mathcal{K}_n(r) dr
$$
\n(2.2) where C is a co-parameters.

No 4\_August 2024| p.26-32<br>where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$  is the Bromwich<br>contour and<br> $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{r^{\alpha} (r^2 + (1 - kr^{\alpha} \sin \alpha \pi))^2}$ . (2.3)

No 4\_August 2024 | p.26-32  
\nwhere 
$$
B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}
$$
 is the Bromwich  
\ncontour and  
\n
$$
\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n k r^{\alpha} \sin \alpha \pi)^2}.
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\n(2.3)

August 2024| p.26-32<br>  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$  is the Bromwich<br>
r and<br>  $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n kr^{\alpha} \sin \alpha \pi)^2}$ . (2.3)<br>  $\psi = (-r + \lambda_n kr^{\alpha} \cos(\alpha \pi) + \lambda_n)$ . Since  $\alpha \in$ <br>
and  $\lambda_n, k > 0$ , we have  $\mathcal{K}_n(r) > 0$ No 4\_August 2024| p.26-32<br>
where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$  is the Bromwich<br>
contour and<br>  $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n k r^{\alpha} \sin \alpha \pi)^2}$ . (2.3)<br>
with  $\psi = (-r + \lambda_n k r^{\alpha} \cos(\alpha \pi) + \lambda_n)$ . Since  $\alpha \in (0,1)$  and  $\lambda_n,$ No 4\_August 2024| p.26-32<br>
where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$  is the Bromwich<br>
contour and<br>  $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n k r^{\alpha} \sin \alpha \pi)^2}$ . (2.3)<br>
with  $\psi = (-r + \lambda_n k r^{\alpha} \cos(\alpha \pi) + \lambda_n)$ . Since  $\alpha \in (0,1)$  and  $\lambda_n,$ No 4\_August 2024| p.26-32<br>
where  $B_r = \{z : \text{Re } z = \sigma, \sigma > 0\}$  is the Bromwich<br>
contour and<br>  $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n k r^{\alpha} \sin \alpha \pi)^2}$ . (2.3)<br>
with  $\psi = (-r + \lambda_n k r^{\alpha} \cos(\alpha \pi) + \lambda_n)$ . Since  $\alpha \in$ <br>
(0, 1) and  $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n k r^{\alpha} \sin \alpha \pi)^2}$ . (2.3)<br>
with  $\psi = (-r + \lambda_n k r^{\alpha} \cos(\alpha \pi) + \lambda_n)$ . Since  $\alpha \in$ <br>
(0,1) and  $\lambda_n, k > 0$ , we have  $\mathcal{K}_n(r) > 0$  for all<br>  $r > 0$ .<br> **Lemma 2.1.** The functions  $\mathcal{S}_n(\alpha, t)$  $\mathcal{K}_n(r) = \frac{k}{\pi} \frac{\lambda_n r^{\alpha} \sin \alpha \pi}{\psi^2 + (\lambda_n k r^{\alpha} \sin \alpha \pi)^2}$ . (2.3)<br>
with  $\psi = (-r + \lambda_n k r^{\alpha} \cos(\alpha \pi) + \lambda_n)$ . Since  $\alpha \in$ <br>
(0, 1) and  $\lambda_n, k > 0$ , we have  $\mathcal{K}_n(r) > 0$  for all<br>  $r > 0$ .<br> **Lemma 2.1.** The functions  $\mathcal{S}_n(\alpha, t$ The functions  $S_n(\alpha, t)$ , and  $\alpha \in$ <br>  $k > 0$ , we have  $K_n(r) > 0$  for all<br>
The functions  $S_n(\alpha, t)$ ,  $n = 1, 2, ...$ ,<br>
wing properties:<br>
1;<br>
sts a constant  $C = C(k, \alpha) > 0$  such<br>  $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1-\alpha}}$ .<br>
by (i) has been proven

 $1)$   $0000$ that

$$
\mathcal{S}_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1 - \alpha}}.
$$

3 have  $\mathcal{K}_n(r) > 0$  for all<br>
ions  $\mathcal{S}_n(\alpha, t)$ ,  $n = 1, 2, ...,$ <br>
ties:<br>
ant  $C = C(k, \alpha) > 0$  such<br>  $\frac{C}{1 + \lambda_n t^{1-\alpha}}$ .<br>
sen proven in (Bazhlekova,<br>
d Property (ii) has been **Lemma 2.1.** The functions  $S_n(\alpha, t)$ ,  $n = 1, 2, ...$ ,<br>have the following properties:<br>(i)  $S_n(\alpha, 0) = 1$ ;<br>(ii) There exists a constant  $C = C(k, \alpha) > 0$  such<br>that<br> $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1-\alpha}}$ .<br>Proof. Property (i) has been proven i **Lemma 2.1.** The functions  $S_n(\alpha, t)$ ,  $n = 1, 2, ...,$ <br>
have the following properties:<br>
(i)  $S_n(\alpha, 0) = 1$ ;<br>
(ii) There exists a constant  $C = C(k, \alpha) > 0$  such<br>
that<br>  $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1 - \alpha}}$ .<br>
Proof. Property (i) has been pr **Lemma 2.1.** The functions  $S_n(\alpha, t)$ ,  $n = 1, 2, \ldots$ ,<br>
have the following properties:<br>
(i)  $S_n(\alpha, 0) = 1$ ;<br>
(ii) There exists a constant  $C = C(k, \alpha) > 0$  such<br>
that<br>  $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1 - \alpha}}$ .<br>
Proof. Property (i) has been From Lemma 2.1(ii), it can be seen that  $\mathcal{P}_{\alpha}(t) \leq \frac{C}{1 + \lambda_n t^{1 - \alpha}}$ .<br>
Proof. Property (i) has been proven in (Bazhlekova,<br>
2015, Theorem 2.2), and Property (ii) has been<br>
proven in (W. J. Zhou Y., 2021, Lemma 3.1). (*i*)  $S_n(\alpha, 0) = 1$ ;<br>
(*ii*) There exists a constant  $C = C(k, \alpha) > 0$  suce<br>
that<br>  $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1 - \alpha}}$ .<br> *Proof.* Property (i) has been proven in (Bazhlekova<br>
2015, Theorem 2.2), and Property (ii) has bee<br>
proven in bounded in  $L^2(\Omega)$  for all  $t \geq 0$ . (*i*)  $S_n(\alpha, 0) = 1$ ;<br>
(*ii*) There exists a constant  $C = C(k, \alpha) > 0$  such<br>
that<br>  $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1-\alpha}}$ .<br> *Proof.* Property (i) has been proven in (Bazhlekova,<br>
2015, Theorem 2.2), and Property (ii) has been<br>
proven i  $\sum_{n=0}^{\infty}$ <br>
be a family of linear<br>
be a family of linear<br>
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en:<br>
us on  $\mathbb{R}_{+}$ . Further- $S_n(\alpha, t) \leq \frac{C}{1 + \lambda_n t^{1-\alpha}}.$ <br> *Proof.* Property (i) has been proven in (Bazhlekova,<br>
2015, Theorem 2.2), and Property (ii) has been<br>
proven in (W. J. Zhou Y., 2021, Lemma 3.1).  $\Box$ <br>
From Lemma 2.1(ii), it can be seen th Proof. Property (i) has been proven in (Bazhlekova,<br>2015, Theorem 2.2), and Property (ii) has been<br>proven in (W. J. Zhou Y., 2021, Lemma 3.1).  $\Box$ <br>From Lemma 2.1(ii), it can be seen that  $\mathcal{P}_{\alpha}(t)$  is<br>bounded in  $L^2(\$ *Proof.* Property (i) has been proven in (Bazhlekova,<br>2015, Theorem 2.2), and Property (ii) has been<br>proven in (W. J. Zhou Y., 2021, Lemma 3.1).  $\Box$ <br>From Lemma 2.1(ii), it can be seen that  $\mathcal{P}_{\alpha}(t)$  is<br>bounded in  $L^$ proven in (W. J. Zhou Y., 2021, Lemma 3.1).  $\Box$ <br>
From Lemma 2.1(ii), it can be seen that  $\mathcal{P}_{\alpha}(t)$  is<br>
bounded in  $L^2(\Omega)$  for all  $t \ge 0$ .<br> **Lemma 2.2.** Let  ${\mathcal{P}_{\alpha}(t)}_{t \ge 0}$  be a family of linear<br>
operators defi

 $[\delta, \infty)$ ; 1 $t \geq 0$ .<br>  $\{t\}\}_{t\geq 0}$  be a family of linear<br>
1). Then:<br>
ntinuous on  $\mathbb{R}_+$ . Further-<br>
e continuity is uniform on<br>
and  $t \geq 0$ ,<br>  $\eta(t - s)AP_{\alpha}(s)v ds$ ,<br>  $t^{-\alpha}$ . operators defined by (2.1). Then:<br>
(i)  $\mathcal{P}_{\alpha}(t)$  is strongly continuous on  $\mathbb{R}_{+}$ . Furthermore, for all  $\delta > 0$ , the continuity is uniform on  $[\delta, \infty)$ ;<br>
(ii) For every  $v \in D(A)$  and  $t \ge 0$ ,<br>  $\mathcal{P}_{\alpha}(t)v = v - \int_{0}^{$ 

$$
\mathcal{P}_{\alpha}(t)v = v - \int_0^t \eta(t - s) A \mathcal{P}_{\alpha}(s)v ds,
$$

where  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}$ .

 $\Gamma(a+b)$   $\qquad \qquad P_{\alpha}(t)v$  and  $AP_{\alpha}(t)v$  are L (i)  $\mathcal{P}_{\alpha}(t)$  is strongly continuous on  $\mathbb{R}_{+}$ . Furthermore, for all  $\delta > 0$ , the continuity is uniform on  $[\delta, \infty)$ ;<br>
(ii) For every  $v \in D(A)$  and  $t \ge 0$ ,<br>  $\mathcal{P}_{\alpha}(t)v = v - \int_{0}^{t} \eta(t - s) A \mathcal{P}_{\alpha}(s)v ds$ ,<br>
where  $\eta(t$  $^{2}(\Omega),$ α(t)<sup>v</sup> and <sup>A</sup>Pα(t)<sup>v</sup> are Lipschitz continuous on more, jor at  $\delta > 0$ , the continuity is anyorm on<br>  $[\delta, \infty)$ ;<br>
(ii) For every  $v \in D(A)$  and  $t \ge 0$ ,<br>  $\mathcal{P}_{\alpha}(t)v = v - \int_0^t \eta(t-s) A \mathcal{P}_{\alpha}(s)v ds$ ,<br>
where  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}$ .<br>
Lemma 2.3. For every  $\delta > 0$  and  $v \in L^2(\Omega$ (*ii*) For every  $v \in D(A)$  and  $t \ge 0$ ,<br>  $\mathcal{P}_{\alpha}(t)v = v - \int_0^t \eta(t-s)AP_{\alpha}(s)v ds$ ,<br>
where  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)}t^{-\alpha}$ .<br> **Lemma 2.3.** For every  $\delta > 0$  and  $v \in L^2(\Omega)$ ,<br>  $\mathcal{P}'_{\alpha}(t)v$  and  $AP_{\alpha}(t)v$  are Lipschitz continuous on<br> **3.** For every  $0 > 0$  and  $v \in L$  ( $\Omega$ ),<br>  $A\mathcal{P}_{\alpha}(t)v$  are Lipschitz continuous on<br>
Moreover, for  $0 < s < t \leq T$ , we have<br>
ag estimates:<br>  $\mathcal{P}'_{\alpha}(t) - \mathcal{P}'_{\alpha}(s) \Vert_{B(L^2(\Omega))} \leq \frac{t-s}{st}$ ,<br>  $\mathcal{P}_{\alpha}(s) \Vert_{B(L^2(\Omega))} \leq C_1 T^{\$  $\mathcal{P}'_{\alpha}(t)v$  and  $A\mathcal{P}_{\alpha}(t)v$  are Lipschitz continuous on<br>  $\delta \leq t \leq T$ . Moreover, for  $0 < s < t \leq T$ , we have<br>
the following estimates:<br>  $\|\mathcal{P}'_{\alpha}(t) - \mathcal{P}'_{\alpha}(s)\|_{B(L^2(\Omega))} \leq \frac{t-s}{st},$ <br>
and<br>  $\|A\mathcal{P}_{\alpha}(t) - A\mathcal{P}_{\alpha}($ 

$$
\|\mathcal{P}'_{\alpha}(t)-\mathcal{P}'_{\alpha}(s)\|_{B(L^2(\Omega))}\leq \frac{t-s}{st},
$$

and

$$
||A\mathcal{P}_{\alpha}(t) - A\mathcal{P}_{\alpha}(s)||_{B(L^{2}(\Omega))} \leq C_{1}T^{\alpha}\frac{t-s}{st},
$$

parameters.

the following estimates:<br>  $\|\mathcal{P}'_{\alpha}(t) - \mathcal{P}'_{\alpha}(s)\|_{B(L^2(\Omega))} \leq \frac{t-s}{st},$ <br>
and<br>  $\|A\mathcal{P}_{\alpha}(t) - A\mathcal{P}_{\alpha}(s)\|_{B(L^2(\Omega))} \leq C_1 T^{\alpha} \frac{t-s}{st},$ <br>
where  $C_1$  is a constant dependent on the problem<br>
parameters.<br> **Lemma 2.4.** F  $\{\mathcal{P}_{\alpha}(t)\}_{t>0}$  defined by (2.1), we have the following  $\begin{aligned} &\mathcal{H}_{\alpha}(t)-\mathcal{P}_{\alpha}'(s)\Vert_{B(L^2(\Omega))} \leq \frac{t-s}{st},\\ &)-A\mathcal{P}_{\alpha}(s)\Vert_{B(L^2(\Omega))} \leq C_1 T^{\alpha} \frac{t-s}{st},\\ &\text{as~constant~dependent~on~the~problem},\\ &\mathbf{4.}~~For~0~<~\gamma~\leq~1,~~for~the~family\ defined~by~(2.1),~we~have~the~following} \end{aligned}$ results:  $||AP_{\alpha}(t) - AP_{\alpha}(s)||_{B(L^{2}(\Omega))} \leq C_{1}T^{\alpha} \frac{t-s}{st},$ <br>
where  $C_{1}$  is a constant dependent on the problem<br>
parameters.<br> **Lemma 2.4.** For  $0 < \gamma \leq 1$ , for the family<br>  $\{\mathcal{P}_{\alpha}(t)\}_{t\geq 0}$  defined by (2.1), we have the follow

$$
||A^{\gamma} \mathcal{P}_{\alpha}(t)||_{B(L^{2}(\Omega))} \leq C t^{(\alpha - 1)\gamma}, \quad t > 0,
$$

parameters.

 $^{2}(\Omega),$ 

*Nguyen Nhu Quan/Vol* 10. N  
*Moreover, for every* 
$$
v \in L^2(\Omega)
$$
,  

$$
\lim_{t \to 0} t^{(1-\alpha)\gamma} ||A^{\gamma} \mathcal{P}_{\alpha}(t)v|| = 0.
$$

Prover, for every 
$$
v \in L^2(\Omega)
$$
,

\n**De**

\n
$$
\lim_{t \to 0} t^{(1-\alpha)\gamma} \|A^{\gamma} \mathcal{P}_{\alpha}(t)v\| = 0.
$$

\n**BA**

\n**BA**

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\n

obtain:

To provide a suitable definition for the solutions of  
\n(1.1), we study the following linear problem:  
\n
$$
\begin{cases}\n\partial_t u + (1 + k \partial_t^\alpha) A u = g(t), t > 0 \\
u(0) = u_0\n\end{cases}
$$
\n(3.1) exponent  $\theta$   
\nBy integrating both sides of the equation (3.1), we  
\nobtain:  
\n
$$
u(t) = u_0 - \int_0^t \eta(t - s) A u(s) ds + \int_0^t g(s) ds
$$
\n(3.2)  
\nNote that  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}$ . Before presenting satisfied. If  
\nthe definition of the integral solution to (3.1), we

(1.1), we study the following linear problem:<br>  $\begin{cases}\n\frac{\partial_t u}{\partial t} + (1 + k\partial_t^{\alpha}) Au = g(t), t > 0 \\
u(0) = u_0\n\end{cases}$  (3.1) export<br>
By integrating both sides of the equation (3.1), we  $\frac{||g(t)||_2}{dt}$ <br>
obtain:<br>  $u(t) = u_0 - \int_0^t \eta(t - s) Au(s) ds + \$ Note that  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}$ . Before presenting<br>the definition of the integral solution to (3.1), we the solution of the solutions of In this paper, we assume t<br>
f linear problem:<br>
(H) The function  $g(t)$  is<br>  $= g(t), t > 0$ <br>
(3.1) exponent  $\theta \in (0, 1)$ , meanit<br>
the equation (3.1), we  $||g(t) - g(s)|| \le L_1 |t - s|^6$ <br>
where  $L_1$  is a co (H) The function  $g(t)$  is Hölder<br>  $\begin{cases} \partial_t u + (1 + k \partial_t^{\alpha}) Au = g(t), t > 0 \ u(0) = u_0 \end{cases}$  (3.1) exponent  $\theta \in (0, 1)$ , meaning that<br>
By integrating both sides of the equation (3.1), we<br>  $||g(t) - g(s)|| \le L_1 |t - s|^{\theta}$ , for a<br>
obtain:<br>  $u(t$  $\begin{cases} \partial_t u + (1 + k \partial_t^\alpha) A u = g(t), t > 0 \qquad (3.1) \end{cases}$  export<br>By integrating both sides of the equation (3.1), we  $\|g(t) \|$ <br>obtain:<br> $u(t) = u_0 - \int_0^t \eta(t - s) A u(s) ds + \int_0^t g(s) ds$  (3.2)<br>Note that  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}$ . Before presentin By integrating both sides of the equation (3.1), w<br>
obtain:<br>  $u(t) = u_0 - \int_0^t \eta(t - s)Au(s)ds + \int_0^t g(s)ds$  (3.2<br>
Note that  $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)}t^{-\alpha}$ . Before presentin<br>
the definition of the integral solution to (3.1), w<br>
state t be that  $\eta(t) = 1 + \frac{\kappa}{\Gamma(1-\alpha)} t^{-\alpha}$ . Before presenting<br>definition of the integral solution to (3.1), we<br>refuse the following lemma.<br> **nma 3.1.** If<br>  $= u_0 - \int_0^t \eta(t-s)Au(s)ds + \int_0^t g(s)ds, t \in [0, T]$ <br>  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \math$ 

Let the following lemma.

\nLemma 3.1. If

\n
$$
w(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - u(t)) = u_0 - \int_0^t \eta(t-s)Au(s)ds + \int_0^t g(s)ds, t \in [0, T]
$$
\nthen

\n
$$
u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds.
$$
\n(3.3)

\n
$$
Proof. \text{ Using the Laplace transform for the equation}
$$
\n
$$
w(t) = \mathcal{P}''_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds.
$$
\n
$$
w(t+h) - w(t)
$$
\n
$$
= \int_0^t (\mathcal{P}'_{\alpha}(t+h-s) - \mathcal{P}'_{\alpha}(t-s))ds + \int_0^t \mathcal{P}_{\alpha}(t-h-s)ds.
$$
\n
$$
= \int_0^t (\mathcal{P}'_{\alpha}(t+h-s) - \mathcal{P}'_{\alpha}(t-s))ds + \int_0^t \mathcal{P}'_{\alpha}(t-h-s)ds.
$$
\n
$$
= \int_0^t \mathcal{P}'_{\alpha}(t+h-s) - \mathcal{P}'_{\alpha}(t-s)ds.
$$
\n
$$
= \int_0^t \mathcal{P}'_{\alpha}(t+h-s) - \mathcal{P}'_{\alpha}(t-s)ds.
$$

then

$$
u(t) = u_0 - \int_0^t \eta(t-s) A u(s) ds + \int_0^t g(s) ds, t \in [0, 1]
$$
  
\nthen  $w(t) \in$   
\n $u(t) = \mathcal{P}_{\alpha}(t) u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.3)  
\nProof. For (4.12)  
\nProof. Using the Laplace transform for the equation  
\n
$$
u(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.
$$
 (3.3)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t+h) g(s) ds$   
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t+h) g(s) ds$   
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t+h) g(s) ds$   
\n $= \int_0^t (\mathcal{P}_{\alpha}'(t-s) g(s) ds)$   
\n $\tilde{u}(t) = \mathcal{P}_{\alpha}(t) \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.3)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.4)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$   
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.5)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.6)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.7)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.8)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.9)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.9)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$  (3.9)  
\n $w(t+h) - \int_0^t \mathcal{P}_{\alpha}(t-s) g(s) ds.$ 

$$
\widehat{u}(\lambda) = \lambda^{-1}u_0 - \widehat{\eta}(\lambda)A\widehat{u}(\lambda) + \lambda^{-1}\widehat{g}(\lambda).
$$

$$
(I + \widehat{\eta}(\lambda)A)\widehat{u}(\lambda) = \lambda^{-1}u_0 + \lambda^{-1}\widehat{g}(\lambda)
$$

$$
u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^{\infty} \mathcal{P}_{\alpha}(t-s)g(s)ds.
$$
 (3.3)  
\n*Proof.* Using the Laplace transform for the equation  
\n(3.2), we obtain:  
\n
$$
\hat{u}(\lambda) = \lambda^{-1}u_0 - \hat{\eta}(\lambda)A\hat{u}(\lambda) + \lambda^{-1}\hat{g}(\lambda).
$$
It means  
\n
$$
(I + \hat{\eta}(\lambda)A)\hat{u}(\lambda) = \lambda^{-1}u_0 + \lambda^{-1}\hat{g}(\lambda)
$$
  
\n
$$
= \int_0^{t} (\mathcal{P}'_{\alpha}(t+h-s) - \mathcal{P}'_{\alpha}(t))dt
$$
  
\nWe have  
\n
$$
\hat{u}(\lambda) = (I + \hat{\eta}(\lambda)A)^{-1}(\lambda^{-1}u_0 + \lambda^{-1}\hat{g}(\lambda))
$$
  
\nWe estimate each component  
\n
$$
= \int_0^{\infty} e^{-\lambda t} (\mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds) dt
$$
  
\nthe proof is complete.  
\n
$$
\Box \qquad \mathcal{P}'_{\alpha}(t)v = \sum_{n=1}^{\infty} \mathcal{S}'_n(\alpha, t) (v, \varphi_n)
$$
  
\nFrom the above discussions, we use the following  
\nconcept of an integral solution for the problem  
\n(3.1).  
\n**Definition 3.1.** The function  $u : [0, T] \rightarrow L^2(\Omega)$   
\n
$$
= \int_0^{\infty} (t+h-s)g(t) \, dt
$$
  
\n
$$
= \int_0^{\infty} \mathcal{S}'_n(\alpha, t) (v, \varphi_n) \, dt
$$
  
\n
$$
= \int_0^{\infty} \mathcal{S}'_n(\alpha, t) (v, \varphi_n) \, dt
$$
  
\n
$$
= \int_0^{\infty} e^{-\lambda t} \left( \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds \right) dt
$$
  
\n
$$
= \int_0^{\infty} \mathcal{P}'_n(t)v = \sum_{n=1}^{\infty} \mathcal{S}'_n
$$

 $(3.1).$  $e^{-\lambda t} \left( \mathcal{P}_{\alpha}(t)u_0 + \int_0^{\infty} \mathcal{P}_{\alpha}(t-s)g(s)ds \right) dt$ <br>proof is complete. [<br>m the above discussions, we use the followin<br>cept of an integral solution for the probler<br>l).<br>finition 3.1. The function  $u : [0, T] \rightarrow L^2(\Omega)$ <br>calle

**Definition 3.1.** The function  $u : [0, T] \rightarrow L^2(\Omega)$  $C([0,T], L^2(\Omega))$  and satisfie

 $=\int_0^\infty e^{-\lambda t} \left( \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds \right) dt$ <br>
the proof is complete.<br>
From the above discussions, we use the following<br>
concept of an integral solution for the problem<br>
(3.1).<br> **Definition 3.1.** The functio  $^{2}(\Omega)$  and  $g \in L^{1}(0,T,L^{2}(\Omega)).$ the proot is complete.<br>
From the above discussions, we use the following<br>
concept of an integral solution for the problem<br>
(3.1).<br> **Definition 3.1.** The function  $u : [0, T] \rightarrow L^2(\Omega)$ <br>
is called an integral solution of (3.1) From the above discussions, we use the following<br>
concept of an integral solution for the problem<br>
(3.1).<br> **Definition 3.1.** The function  $u : [0, T] \rightarrow L^2(\Omega)$ <br>
is called an integral solution of (3.1) if  $u \in [P'_\alpha(t+h-s)(g(s)-\frac{C}{$ concept of an integral solution for the problem<br>
(3.1).<br> **Definition 3.1.** The function  $u : [0, T] \rightarrow L^2(\Omega)$ <br>
is called an integral solution of (3.1) if  $u \in$ <br>  $||\mathcal{P}'_{\alpha}(t+h-s)(g(s) - \mathcal{E}'(0, T], L^2(\Omega))$  and satisfies (3.3).<br>
Ass (3.1).<br> **Definition 3.1.** The function  $u : [0, T] \rightarrow L^2(\Omega)$ <br>
is called an integral solution of (3.1) if  $u \in$ <br>  $C([0, T], L^2(\Omega))$  and satisfies (3.3).<br>
Assume that  $u_0 \in L^2(\Omega)$  and  $g \in L^1(0, T, L^2(\Omega))$ .<br>
Lemma 3.1 shows that the

*Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.20<br> *Moreover, for every*  $v \in L^2(\Omega)$ , **Definition 3.2.** The<br>  $\lim_{t\to 0} t^{(1-\alpha)\gamma} ||A^{\gamma}P_{\alpha}(t)v|| = 0$ .<br>
3 MAIN RESULTS<br>
To provide a suitable definition for the solutions of<br>
Th *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br> *Moreover, for every*  $v \in L^2(\Omega)$ ,<br> **Definition 3.2.** The function of<br>  $\lim_{t\to 0} t^{(1-\alpha)\gamma} ||A^{\gamma}P_{\alpha}(t)v|| = 0$ .<br>  $L^2(\Omega)$  is called a classical solution  $u \in C([0, T]; L^2(\Omega))$ *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.<br> *Moreover, for every*  $v \in L^2(\Omega)$ ,<br> **Definition 3.2.** Th<br>  $\lim_{t\to 0} t^{(1-\alpha)\gamma} ||A^{\gamma} \mathcal{P}_{\alpha}(t)v|| = 0$ .<br>  $L^2(\Omega)$  is called a cl<br>  $u \in C([0, T]; L^2(\Omega))$ <br>
3 MAIN RESULTS<br>  $u(t) \in D(A)$  er, for every  $v \in L^2(\Omega)$ ,<br>  $\lim_{t \to 0} t^{(1-\alpha)\gamma} ||A^{\gamma} \mathcal{P}_{\alpha}(t)v|| = 0.$ <br>  $L^2(\Omega)$  is called a cl<br>  $u \in C([0, T]; L^2(\Omega))$ <br>  $u(t) \in D(A)$  for  $u(t) \in D(A)$  for  $\Omega$ <br>
NRESULTS<br>
ide a suitable definition for the solutions of<br>
e study the  $\lim_{t\to 0} t^{(1-\alpha)\gamma} ||A^{\gamma} \mathcal{P}_{\alpha}(t)v|| = 0.$   $u \in C([0, T]; L^{2}(\Omega))$  with  $\partial_{t} u \in C$ <br>
3 MAIN RESULTS  $C((0, T]; L^{2}(\Omega))$ , and satisfies (3.<br>
To provide a suitable definition for the solutions of  $(1.1)$ , we study the following line No 4\_August 2024| p.26-32<br> **Definition 3.2.** The function  $u : [0, T] \rightarrow$ <br>  $L^2(\Omega)$  is called a classical solution of (3.1) if<br>  $u \in C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega))$ ,<br>  $u(t) \in D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u \in$  $L^2(\Omega)$  is called a classic No 4\_August 2024| p.26-32<br> **Definition 3.2.** The function  $u : [0, T] \rightarrow$ <br>  $L^2(\Omega)$  is called a classical solution of (3.1) if<br>  $u \in C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega)),$ <br>  $u(t) \in D(A)$  for all  $t \in (0, T], \ \partial_t^{\alpha} A u \in$  $([0,T];L^2(\Omega))$  with  $\partial_t u \in C((0,T];L^2(\Omega)),$ ugust 2024| p.26-32<br>
ion 3.2. The function  $u : [0, T] \rightarrow$ <br>
is called a classical solution of (3.1) if<br>  $[0, T]; L^2(\Omega)$  with  $\partial_t u \in C((0, T]; L^2(\Omega)),$ <br>  $D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u \in$ <br>  $[0, T]; L^2(\Omega)$ , and satisfies (3.1). :  $[0, T] \rightarrow$ <br>
m of (3.1) if<br>  $(0, T]; L^2(\Omega)),$ <br>
T],  $\partial_t^{\alpha} A u \in$ , No 4\_August 2024| p.26-32<br>
Definition 3.2. The function  $u : [0, T] \rightarrow$ <br>  $L^2(\Omega)$  is called a classical solution of (3.1) if<br>  $u \in C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega)),$ <br>  $u(t) \in D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u \in$ <br>  $C((0, T]; L^2(\Omega$  $u(t) \in D(A)$  for all  $t \in (0,T], \partial_t^{\alpha}Au \in$  $C((0, T]; L^2(\Omega))$ , and satisfies (3. 4\_August 2024| p.26-32<br> **finition 3.2.** The function  $u : [0, T]$ <br>
(2) is called a classical solution of (3.1<br>  $\vdots C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega)) \in D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u$ <br>
(0, T];  $L^2(\Omega)$ ), and satisfies (3 No 4\_August 2024| p.26-32<br> **Definition 3.2.** The function  $u : [0, T] \rightarrow$ <br>  $L^2(\Omega)$  is called a classical solution of (3.1) if<br>  $u \in C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega)),$ <br>  $u(t) \in D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u \in$ <br>  $C((0, T]; L^2$ No 4\_August 2024| p.26-32<br> **Definition 3.2.** The function  $u : [0, T] \rightarrow$ <br>  $L^2(\Omega)$  is called a classical solution of (3.1) if<br>  $u \in C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega))$ ,<br>  $u(t) \in D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u \in$ <br>  $C((0, T]; L^$ **Definition 3.2.** The function  $u : [0, T] \rightarrow$ <br>  $L^2(\Omega)$  is called a classical solution of (3.1) if<br>  $u \in C([0, T]; L^2(\Omega))$  with  $\partial_t u \in C((0, T]; L^2(\Omega)),$ <br>  $u(t) \in D(A)$  for all  $t \in (0, T], \partial_t^{\alpha} A u \in$ <br>  $C((0, T]; L^2(\Omega)),$  and satisfies (3.1).<br> all solution of  $(3.1)$  if<br>  $\partial_t u \in C((0,T]; L^2(\Omega)),$ <br>  $\in (0,T], \partial_t^{\alpha} A u \in$ <br>
sfies  $(3.1).$ <br>
the following:<br>
Hölder continuous with<br>
ng that<br>
, for all  $0 < s, t \leq T$ ,

exponent  $\theta \in (0,1)$ , meaning that  $u(t) \in D(A)$  for all  $t \in (0, T]$ ,  $\partial_t^T Au \in$ <br>  $C((0, T]; L^2(\Omega))$ , and satisfies (3.1).<br>
In this paper, we assume the following:<br> **(H)** The function  $g(t)$  is Hölder continuous with<br>
exponent  $\theta \in (0, 1)$ , meaning that<br>  $||g(t) - g(s)|| \$ In this paper, we assume the following:<br>
(**H**) The function  $g(t)$  is Hölder continuous with<br>
exponent  $\theta \in (0, 1)$ , meaning that<br>  $||g(t) - g(s)|| \le L_1|t - s|^{\theta}$ , for all  $0 < s, t \le T$ ,<br>
where  $L_1$  is a constant.<br> **Lemma 3.2.** Assu In this paper, we assume the following:<br>
(H) The function  $g(t)$  is Hölder continuous with<br>
exponent  $\theta \in (0, 1)$ , meaning that<br>  $||g(t) - g(s)|| \le L_1 |t - s|^{\theta}$ , for all  $0 < s, t \le T$ ,<br>
where  $L_1$  is a constant.<br>
Lemma 3.2. Assume t

$$
||g(t) - g(s)|| \le L_1|t - s|^{\theta}
$$
, for all  $0 < s, t \le T$ ,

**Lemma 3.2.** Assume that the assumption  $(H)$  is

equation (3.1), we

\n
$$
||g(t) - g(s)|| \le L_1 |t - s|^{\theta}, \quad \text{for all } 0 < s, t \le T,
$$
\nwhere  $L_1$  is a constant.

\n
$$
\int_0^t g(s)ds \quad (3.2)
$$
\nLemma 3.2. Assume that the assumption  $(H)$  is satisfied. If

\ntion to (3.1), we

\n
$$
w(t) = \int_0^t \mathcal{P}'_{\alpha}(t - s)(g(s) - g(t)) ds, \quad t \in (0, T],
$$
\n
$$
g(s)ds, t \in [0, T]
$$
\nthen  $w(t) \in C^{\theta}((0, T]; L^2(\Omega))$ .

\nProof. For  $0 < t < t + h \le T$ , we have:

\n
$$
w(t + h) - w(t)
$$
\nfor the equation

\n
$$
f^t
$$

**Lemma 3.1.** 
$$
f \qquad w(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t)) ds, \quad t \in (0, T],
$$
\n
$$
u(t) = u_0 - \int_0^t \eta(t-s)Au(s)ds + \int_0^t g(s)ds, t \in [0, T]
$$
\nthen 
$$
u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds.
$$
\n(3.3)\nProof. Using the Laplace transform for the equation\n
$$
u(t) = \int_0^t (\mathcal{P}'_{\alpha}(t+h) - w(t)) ds + \int_0^t (\mathcal{P}'_{\alpha}(t+h) - w(t)) ds + \int_0^t (\mathcal{P}'_{\alpha}(t+h) - w(t)) ds + \int_0^t (\mathcal{P}'_{\alpha}(t+h) - s) - \mathcal{P}'_{\alpha}(t-s)) (g(s) - g(t)) ds + \int_0^t (\mathcal{P}'_{\alpha}(t+h) - s) - \mathcal{P}'_{\alpha}(t-s)) ds + \int_0^t (\mathcal{P}'_{\alpha}(t+h) - s) - \mathcal{P}'_{\alpha}(t-s)) ds + \int_0^t \mathcal{P}'_{\alpha}(t+h) - \int_0^t (\mathcal{P}'_{\alpha}(t+h) - s) - \mathcal{P}'_{\alpha}(t-s)) ds + \int_0^t \mathcal{P}'_{\alpha}(t+h) - \int_0^t (\mathcal{P}'_{\alpha}(t+h) - s) - \mathcal{P}'_{\alpha}(t-s)) ds + \int_0^t \mathcal{P}'_{\alpha}(t+h) - \int_0^t (\mathcal{P}'_{\alpha}(t+h) - s) - \int_0^t (\mathcal{P}'_{\alpha}(t) - s) - \int_0^t
$$

nents separately. For  $I_1$ , based on

$$
=I_1 + I_2 + I_3
$$
\n
$$
(\sqrt{-1}\hat{g}(\lambda))
$$
\nWe estimate each component in the three components separately. For  $I_1$ , based on\n
$$
\Box \qquad \mathcal{P}'_{\alpha}(t)v = \sum_{n=1}^{\infty} \mathcal{S}'_n(\alpha, t) (v, \varphi_n) \varphi_n, \|\mathcal{P}'_{\alpha}(t)\|_{B(L^2(\Omega))}
$$
\nthe following\nthe problem\n
$$
\leq \frac{1}{t}.
$$
\n(3.4)\n
$$
(\mathcal{T}^1) \to L^2(\Omega)
$$
\nWith  $t > 0$  and (H), we have:\n
$$
\|\mathcal{P}'_{\alpha}(t + h - s)(g(s) - g(t))\|
$$
\n
$$
\leq L_1(t + h - s)^{-1}(t - s)^{\theta}
$$
\n(0,  $T, L^2(\Omega)$ ).\n
$$
\leq L_1(t - s)^{\theta - 1}
$$
\n(3.1) has a\nreested in im- for all  $0 < h \leq T - t$ , the Dominated Conventate the inte-  
\nvertex Theorem (Lebesgue's Dominated Conver-

With  $t > 0$  and (H), we have:

$$
\|\mathcal{P}_{\alpha}'\ (t+h-s)(g(s)-g(t))\|
$$
  
\n
$$
\leq L_1(t+h-s)^{-1}(t-s)^{\theta}
$$
  
\n
$$
\leq L_1(t-s)^{\theta-1}
$$

With  $t > 0$  and (H), we have:<br>  $\|\mathcal{P}'_{\alpha}(t+h-s)(g(s) - g(t))\|$ <br>  $\leq L_1(t+h-s)^{-1}(t-s)^{\theta}$ <br>  $\leq L_1(t-s)^{\theta-1}$ <br>
for all  $0 < h \leq T - t$ , the Dominated Con-<br>
vergence Theorem (Lebesgue's Dominated Convergence Theorem) implies that  $I_1 \to$ Vith  $t > 0$  and (H), we have:<br>  $\|\mathcal{P}'_{\alpha}(t+h-s)(g(s)-g(t))\|$ <br>  $\leq L_1(t+h-s)^{-1}(t-s)^{\theta}$ <br>  $\leq L_1(t-s)^{\theta-1}$ <br>
for all  $0 < h \leq T-t$ , the Dominated Convergence Theorem (Lebesgue's Dominated Convergence Theorem) implies that  $I_1 \to 0$  as With  $t > 0$  and (H), we have:<br>  $\|\mathcal{P}'_{\alpha}(t+h-s)(g(s) - g(t))\|$ <br>  $\leq L_1(t+h-s)^{-1}(t-s)^{\theta}$ <br>  $\leq L_1(t-s)^{\theta-1}$ <br>
for all  $0 < h \leq T - t$ , the Dominated Con-<br>
vergence Theorem (Lebesgue's Dominated Convergence Theorem) implies that  $I_1 \rightarrow$ 

$$
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$$
\n\nFurthermore, using Lemma 2.3, we obtain: \n
$$
||I_1|| \leq \int_0^t ||(\mathcal{P}'_a(t+h-s)) - \mathcal{P}'_a(t-s)) (g(s) - g(t))|| ds
$$
\nthen: \n
$$
-\mathcal{P}'_a(t-s) (g(s) - g(t))|| ds
$$
\n
$$
\leq h \int_0^t \frac{1}{(t+h-s)(t-s)} ||(g(s) - g(t))|| ds
$$
\n
$$
\leq L_1 h \int_0^t \frac{s^{\theta-1}}{s+h} ds
$$
\n
$$
\leq L_1 \int_0^h \frac{h}{s+h} s^{\theta-1} ds + L_1 h \int_h^\infty \frac{s}{s+h} s^{\theta-2} ds
$$
\n
$$
= \int_0^t \frac{\mathcal{P}_\alpha(t)}{t+h} ds
$$
\n
$$
\leq L_1 \frac{h^\theta}{\theta(1-\theta)}.
$$
\nSimilarly,  $I_3$  is estimated as follows: \nFrom (3.4) and (H), the  $g(t)$ ) ds is absolutely the Dominated Convergen

$$
\leq L_1 \frac{1}{\theta(1-\theta)}.
$$
  
\nSimilarly,  $I_3$  is estimated as follows:  
\n
$$
||I_3|| \leq \int_t^{t+h} (t+h-s)^{-1}||(g(s) - g(t+h))||ds
$$
 From (3.4) and (H), the integral  $\int_0^t f(t) ds$  is absolutely convergent  
\n
$$
\leq L_1 \int_t^{t+h} (t+h-s)^{\theta-1} ds
$$
 In the Dominated Convergence Theorem  
\n
$$
\leq L_1 \frac{h^{\theta}}{\theta}
$$
  
\n
$$
\leq L_1 \frac{h^{\theta}}{\theta}
$$
  
\nThe boundedness of  $\mathcal{P}_{\alpha}(t)$  implies that:  
\n
$$
||I_2|| = ||(\mathcal{P}_{\alpha}(t+h) - \mathcal{P}_{\alpha}(h)) (g(t) - g(t+h))||
$$
 From Lemma 2.2(i), we have:  
\n
$$
\leq 2CL_1 h^{\theta}
$$
  
\nBy combining the estimates for  $I_1$ ,  $I_2$ , and  $I_3$ , we  
\nobtain the desired result.  
\nNow we prove the main result of this paper in the  
\nfollowing theorem:  
\n
$$
= \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(s)g(t) ds - \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(s)g(t) ds - \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(s)g(t) ds - \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(s)g(t) ds
$$

$$
\leq L_1 \frac{h^2}{\theta}
$$
  
The boundedness of  $\mathcal{P}_{\alpha}(t)$  implies that:  

$$
||I_2|| = ||(\mathcal{P}_{\alpha}(t+h) - \mathcal{P}_{\alpha}(h)) (g(t) - g(t+h))||
$$

$$
\leq 2CL_1 h^{\theta}
$$
  
By combining the estimates for  $I_1$ ,  $I_2$ , and  $I_3$ , we  
obtain the desired result.  $\square$   
Now we prove the main result of this paper in the  
following theorem:  
**Theorem 3.3.** Let  $u_0 \in L^2(\Omega)$  and  $(H)$  be sat  
isfield. Then, the integral solution of (3.1) is

The boundedness of  $\mathcal{P}_{\alpha}(t)$  implies that:<br>  $||I_2|| = ||(\mathcal{P}_{\alpha}(t+h) - \mathcal{P}_{\alpha}(h)) (g(t) - g(t+h))||$ <br>  $\leq 2CL_1 h^{\theta}$ <br>
By combining the estimates for  $I_1$ ,  $I_2$ , and  $I_3$ , we<br>
obtain the desired result.  $\square$ <br>
Now we prove the m **Theorem 3.3.** Let  $u_0 \in L^2(\Omega)$  and (H) be sat*classical solution. Moreover, we have*  $\partial_t u, \partial_t^{\alpha} Au \in$  By combining these estimates and using the iden- $C^{\theta}\left((0,T];L^2(\Omega)\right).$ combining the estimates for  $I_1$ ,  $I_2$ , and  $I_3$ ,<br>in the desired result.<br>we prove the main result of this paper in<br>wing theorem:<br>**orem 3.3.** Let  $u_0 \,\in L^2(\Omega)$  and (H) be s<br>d. Then, the integral solution of (3.1) i<br>sic . By combining the estimates for  $I_1$ ,  $I_2$ , and  $I_3$ , we<br>
obtain the desired result.<br>
Now we prove the main result of this paper in the<br>
following theorem:<br>  $= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s)g(t)ds$ <br> **Theorem 3.3.** Let  $u_0 \in L$ (5) obtain the desired result.<br>
Now we prove the main result of this paper in the<br>
following theorem:<br> **Theorem 3.3.** Let  $u_0 \,\in L^2(\Omega)$  and (H) be sat-<br>
isfied. Then, the integral solution of (3.1) is a<br>
classical soluti  $u_0 \in L^2(\Omega)$  and (*H*) be sat-<br>  $u_0 \in L^2(\Omega)$  and (*H*) be sat-<br>  $u_0 \in L^2(\Omega)$  and (*H*) be sat-<br>  $u_0 \neq \emptyset$   $\rightarrow (\mathcal{P}_{\alpha}(t) - I) g(t)$  khi  $h \rightarrow$ <br>  $u_0$  are solution of (3.1) is a<br>  $u_0$  be sat-<br>  $u_0 \neq \emptyset$   $\rightarrow (\mathcal{P}_{\alpha}(t) - I$ Let  $u_0 \,\in L^2(\Omega)$  and  $(H)$  be sat<br>  $e$  integral solution of (3.1) is a<br>  $u$ . Moreover, we have  $\partial_t u, \partial_t^{\alpha} Au \in$  By combining these estimates and<br>  $u$ . Moreover, we have  $\partial_t u, \partial_t^{\alpha} Au \in$  By combining these estimates and

$$
u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds, t \in [0, T]
$$

With  $u_0 \in L^2(\Omega)$ , Lemma 2.2(ii) imp *ssped. Then, the integral solution of* (3.1) *is a*<br> *classical solution. Moreover, we have*  $\partial_t u, \partial_t^{\alpha} A u \in$  By combining these estimates an<br>  $C^{\theta} ((0,T]; L^2(\Omega)).$ <br> *Proof.* Let *u* be the integral solution of the problem classical solution. Moreover, we have  $\partial_t u, \partial_t^{\alpha} Au \in$ <br>  $C^{\theta} ((0, T]; L^2(\Omega)).$ <br>
Proof. Let u be the integral solution of the problem<br>
(3.1). Note that<br>  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds, t \in [0, T]$ <br>
With  $u_0 \in L^2(\Omega)$ be the integral solution of the problem<br>
hat<br>  $(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds, t \in [0, T]$ <br>  $L^2(\Omega)$ , Lemma 2.2(ii) implies that<br>  $t > 0$ , is the classical solution of the<br>  $t > 0$ , is the classical solution of the<br>  $\frac{1}{h} \int_0$ Proof. Let u be the integral solution of the problem<br>
(3.1). Note that<br>  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds, t \in [0, T]$ <br>
With  $u_0 \in L^2(\Omega)$ , Lemma 2.2(ii) implies that<br>  $\mathcal{P}_{\alpha}(t)u_0$ , for  $t > 0$ , is the classical *L*<sup>2</sup>(Ω), Lemma 2.2(ii) implies that More<br>  $t > 0$ , is the classical solution of the<br>
oblem:<br>  $\frac{1}{h}$ <br>  $\partial_t u = -(1 + k \partial_t^{\alpha}) Au, t > 0$ <br>  $u(0) = u_0$ <br>
I that<br>  $\Phi(t) = \int_0^t \mathcal{P}_{\alpha}(t - s)g(s)ds$  Than<br>
I solution of the following prob with  $u_0 \t L^2(\Omega)$ , Lemma 2.2(ii) implies that<br>  $\mathcal{P}_{\alpha}(t)u_0$ , for  $t > 0$ , is the classical solution of the<br>
following problem:<br>  $\begin{cases} \frac{\partial_t u}{\partial t} = -(1 + k\partial_t^{\alpha}) A u, t > 0 \\ u(0) = u_0 & \frac{1}{h} \int_0^h \mathcal{P}_{\alpha}(s) g(t + h - h) \\ u(0) = u_0 & \frac$ 

$$
\begin{cases} \partial_t u = -(1 + k \partial_t^{\alpha}) Au, t > 0 \\ u(0) = u_0 \end{cases}
$$

$$
\Phi(t) = \int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds
$$

ng problem:  
\n
$$
\frac{1}{h}
$$
\n
$$
\begin{cases}\n\partial_t u = -(1 + k\partial_t^{\alpha}) Au, t > 0 \\
u(0) = u_0\n\end{cases}
$$
\nrecall that  
\n
$$
\Phi(t) = \int_0^t \mathcal{P}_{\alpha}(t - s) g(s) ds
$$
\n
$$
\text{snical solution of the following problem:} \quad \text{have:} \quad \begin{cases}\n\partial_t u + (1 + k\partial_t^{\alpha}) Au = g(t), t > 0 \\
u(0) = 0\n\end{cases} \quad \begin{cases}\n\frac{1}{h} \int_0^t u(t) dt = 0 \\
\frac{1}{h} \int_0^t u(t) dt = 0\n\end{cases}
$$

No 4\_August 2024| p.26-32<br>For a fixed  $t \in (0, T]$ , we prove that  $\Phi(t)$  is con-<br>tinuously differentiable at  $t$ . Let  $0 < h \leq T - t$ ,<br>then: No 4\_August 2024| p.26-32<br>
For a fixed  $t \in (0, T]$ , we prove that  $\Phi(t)$  is continuously differentiable at  $t$ . Let  $0 < h \leq T - t$ ,<br>
then:<br>  $\frac{\Phi(t+h) - \Phi(t)}{\Phi(t+h)} = \frac{1}{\pi} \left( \int_{0}^{t+h} P_{\infty}(t+h-s) g(s) ds \right)$ 

$$
(s) - g(t)||ds
$$
  
\n
$$
\frac{\Phi(t+h) - \Phi(t)}{h} = \frac{1}{h} \left( \int_0^{t+h} \mathcal{P}_{\alpha}(t+h-s)g(s)ds \right)
$$
  
\n
$$
-\int_0^t \mathcal{P}_{\alpha}(t-s)g(s)ds
$$
  
\n
$$
\int_h^\infty \frac{s}{s+h} s^{\theta-2} ds
$$
  
\n
$$
= \int_0^t \frac{\mathcal{P}_{\alpha}(t+h-s) - \mathcal{P}_{\alpha}(t-s)}{h} g(s)ds
$$
  
\n
$$
+ \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(t+h-s)g(s)ds
$$
  
\n
$$
\text{From (3.4) and (H), the integral } \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t+h))|ds \quad \text{is absolutely convergent. By applying the Dominated Convergence Theorem (Lebesgue's Dominated Convergence Theorem), we obtain:}
$$
  
\n
$$
\lim_{h \to 0} \int_0^t \frac{\mathcal{P}_{\alpha}(t+h-s) - \mathcal{P}_{\alpha}(t-s)}{h} (g(s) - g(t))ds
$$

the Dominated Convergence Theorem (Lebesgue's Dominated Convergence Theorem), we obtain: From (3.4) and (H), the integral  $\int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s)-g(t)) ds$  is absolutely convergent. By applying

From (3.4) and (H), the integral 
$$
\int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t)) ds
$$
 is absolutely convergent. By applying  
the Dominated Convergence Theorem (Lebesgue's  
Dominated Convergence Theorem), we obtain:  

$$
\lim_{h\to 0} \int_0^t \frac{\mathcal{P}_{\alpha}(t+h-s) - \mathcal{P}_{\alpha}(t-s)}{h} (g(s) - g(t)) ds
$$

$$
= \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t)) ds
$$
From Lemma 2.2(i), we have:
$$
\int_0^t \frac{\mathcal{P}_{\alpha}(t+h-s) - \mathcal{P}_{\alpha}(t-s)}{h} g(t) ds
$$

$$
\lim_{h\to 0} \int_{0}^{t} \frac{\sinh(\theta)}{h} = \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s) \cdot \frac{r_{\alpha}(t-s)}{h} (g(s) - g(t)) ds
$$
\n
$$
\leq L_{1} \frac{h^{\theta}}{\theta} = \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s) (g(s) - g(t)) ds
$$
\n
$$
||I_{2}|| = ||(\mathcal{P}_{\alpha}(t+h) - \mathcal{P}_{\alpha}(h)) (g(t) - g(t+h))||
$$
\nFrom Lemma 2.2(i), we have:  
\n
$$
\leq 2CL_{1}h^{\theta}
$$
\nBy combining the estimates for  $I_{1}$ ,  $I_{2}$ , and  $I_{3}$ , we obtain the desired result.  
\nNow we prove the main result of this paper in the following theorem:  
\n
$$
= \frac{1}{h} \int_{h}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac{1}{h} \int_{0}^{t} \mathcal{P}_{\alpha}(s) g(t) ds
$$
\n
$$
= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac{1}{h} \int_{0}^{h} \mathcal{P}_{\alpha}(s) g(t) ds
$$
\n
$$
= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac{1}{h} \int_{0}^{h} \mathcal{P}_{\alpha}(s) g(t) ds
$$
\n
$$
= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac{1}{h} \int_{0}^{h} \mathcal{P}_{\alpha}(s) g(t) ds
$$
\n
$$
= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac{1}{h} \int_{0}^{h} \mathcal{P}_{\alpha}(s) g(t) ds
$$
\n
$$
= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac{1}{h} \int_{0}^{h} \mathcal{P}_{\alpha}(s) g(t) ds
$$
\n
$$
= \frac{1}{h} \int_{t}^{t+h} \mathcal{P}_{\alpha}(s) g(t) ds - \frac
$$

tity  $\int_0^t \mathcal{P}'_{\alpha}(t-s)g(s) ds = (\mathcal{P}_{\alpha}(t) - I) g(t)$ , we obtain:

$$
\Rightarrow (\mathcal{P}_{\alpha}(t) - I) g(t) \text{ kln } h \rightarrow 0
$$
\ncombining these estimates and using the iden-

\n
$$
\int_0^t \mathcal{P}'_{\alpha}(t - s) g(s) ds = (\mathcal{P}_{\alpha}(t) - I) g(t), \text{ we ob-}
$$
\n
$$
\lim_{h \to 0} \int_0^t \frac{\mathcal{P}_{\alpha}(t + h - s) - \mathcal{P}_{\alpha}(t - s)}{h} g(s) ds
$$
\n
$$
= \int_0^t \mathcal{P}'_{\alpha}(t - s) g(s) ds
$$
\ncover

\n
$$
\int_0^h \mathcal{P}_{\alpha}(s) g(t + h - s) ds =
$$

Moreover

$$
\begin{aligned}\n &\text{(a)} \quad b_0 \\
 &\text{(b)} \quad c_0 \\
 &\text{(c)} \quad c_1 \\
 &\text{(d)} \quad c_2 \\
 &\text{(e)} \quad c_3 \\
 &\text{(f)} \quad c_4 \\
 &\text{(g)} \quad c_5 \\
 &\text{(h)} \quad c_6 \\
 &\text{(i)} \quad c_7 \\
 &\text{(j)} \quad c_8 \\
 &\text{(k)} \quad c_9 \\
 &\text{(l)} \quad c_9 \\
 &\text{(m)} \quad c_1 \\
 &\text{(n)} \quad c_2 \\
 &\text{(o)} \quad c_3 \\
 &\text{(o)} \quad c_4 \\
 &\text{(h)} \quad c_5 \\
 &\text{(i)} \quad c_6 \\
 &\text{(ii)} \quad c_7 \\
 &\text{(iii)} \quad c_8 \\
 &\text{(iv)} \quad c_9 \\
 &\text{(v)} \quad c_9 \\
$$

have:

$$
\left\|\frac{1}{h}\int_0^h \mathcal{P}_{\alpha}(s)(g(t+h-s)-g(t))ds\right\| \leq CL_1h^{\theta} \to 0,
$$

Nguyen Nhu Quan/Vol 10. No 4\_August 2024| p.26-32<br>
as  $h \to 0$ . From Lemma 2.2(i), we deduce that From Lemma 2.4 and (H), it is eatlier<br>  $\lim_{h\to 0} \frac{1}{h} \int_0^h \mathcal{P}_{\alpha}(s)g(t) ds = g(t)$ . Therefore<br>  $||A\Phi_2(t+h) - A\Phi_2(t)||$  $\lim_{h\to 0} \frac{1}{h} \int_0^h \mathcal{P}_{\alpha}(s)g(t) ds = g(t)$ . Then *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024|<br>
0. From Lemma 2.2(i), we deduce that From Lemma 2.4 and<br>  $\int_0^h \mathcal{P}_{\alpha}(s)g(t) ds = g(t)$ . Therefore<br>  $||A\Phi_2(t+h) - A||_1^2$ <br>  $\int_0^{t+h} \mathcal{P}_{\alpha}(t+h-s)g(s)ds = g(t)$ . Nguyen Nhu Quan/Vol 10. No 4\_Aug<br>
as  $h \to 0$ . From Lemma 2.2(i), we deduce that From Len<br>  $\lim_{h\to 0} \frac{1}{h} \int_0^h \mathcal{P}_{\alpha}(s)g(t) ds = g(t)$ . Therefore<br>  $||A\Phi_2||$ <br>  $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(t+h-s)g(s)ds = g(t)$ .  $\leq C$ <br>
This mea

$$
\Rightarrow 0. \text{ From Lemma 2.2.1}, we denote that
$$
\n
$$
\Rightarrow \int_0^h \int_0^h \mathcal{P}_{\alpha}(s)g(t) \, ds = g(t). \text{ Therefore}
$$
\n
$$
\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \mathcal{P}_{\alpha}(t+h-s)g(s)ds = g(t).
$$
\nmeans that  $\Phi(t)$  is continuously differentiable and its derivative, denoted  $\Phi'_+(t)$ , satisfies:

\n
$$
\Phi'_+(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)g(s)ds + g(t).
$$
\nreasoning as above, assuming that  $0 < h < t$ , and easily deduce that  $\Phi(t)$  is differentiable at

$$
\Phi_+'(t) = \int_0^t \mathcal{P}'_\alpha(t-s)g(s)ds + g(t)
$$

This means that  $\Phi(t)$  is continuously differentiable<br>
at t and its derivative, denoted  $\Phi'_+(t)$ , satisfies:<br>  $\Phi'_+(t) = \int_0^t \mathcal{P}'_a(t-s)g(s)ds + g(t)$ .<br>
By reasoning as above, assuming that  $0 < h < t$ , fore,  $A\Phi_2 \in C$  ( $(0, T]$ ; This means that  $\Phi(t)$  is continuously differe<br>at t and its derivative, denoted  $\Phi'_{+}(t)$ , satis<br> $\Phi'_{+}(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)g(s)ds + g(t)$ .<br>By reasoning as above, assuming that  $0 <$ <br>we can easily deduce that  $\Phi(t)$  is diffe

$$
\Phi'_{+}(t) = \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s)g(s)ds + g(t).
$$
\nBy reasoning as above, assuming that  $0 < h < t$ ,  
\nwe can easily deduce that  $\Phi(t)$  is differentiable at  
\n $t_{-}$  and  $\Phi'_{+}(t) = \Phi'_{-}(t)$ . From Lemma 3.2 and (H),  
\nwe obtain:  
\n
$$
\Phi'(t) = \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s)g(s)ds + g(t)
$$
\n
$$
= \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s)g(s)ds + g(t)
$$
\n
$$
= \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s)g(t)ds + g(t)
$$
\n
$$
= \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s)g(t)ds + g(t)
$$
\nHere, \* denotes the convolution  
\n
$$
+ \int_{0}^{t} \mathcal{P}'_{\alpha}(t-s)g(t)ds + g(t)
$$
\n
$$
= \Phi'(t_{-})
$$
\n
$$
= \Phi'(t_{-})
$$
\nNow we show that  $\Phi(t) \in D(A)$  for all  $0 < t \leq T$ .  
\nTo achieve this, we rewrite:  
\n
$$
A\Phi(t) = A\Phi_{1}(t) + A\Phi_{2}(t)
$$
\n
$$
= \int_{0}^{t} A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds
$$
\n
$$
= \int_{0}^{t} A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds
$$
\n
$$
= \int_{0}^{t} A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds
$$
\n
$$
= \Phi'(t_{-})
$$
\n
$$
= \int_{0}^{t} A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds
$$
\n
$$
= \Phi'(t_{-})
$$
\n
$$
= \int_{0}^{t} A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds
$$
\n
$$
= \Phi'(t_{-})
$$

Now we show that 
$$
\Phi(t) \in D(A)
$$
 for all  $0 < t \leq T$ .  
\nTo achieve this, we rewrite:  
\n
$$
A\Phi(t) = A\Phi_1(t) + A\Phi_2(t)
$$
\n
$$
= \int_0^t A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds
$$
\n
$$
+ \int_0^t A\mathcal{P}_{\alpha}(t-s)g(t)ds
$$
\nFor a fixed  $t \in (0, T]$ , from Lemma 2.4 and  
\n(H), these two integrals converge absolutely. There-  
\nfore  $\Phi(t) \in D(A)$ . By repeating the reasoning  
\nused in the proof of Lemma 2.2, we have  $A\Phi_1 \in$   
\n
$$
A\Phi(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds + \mathcal{P}_{\alpha}(t)g(t) \in C^{\alpha}
$$
\n
$$
+ \mathcal{P}_{\alpha}(t)g(t) \in C^{\beta}
$$
\

To achieve this, we rewrite:<br>  $A\Phi(t) = A\Phi_1(t) + A\Phi_2(t)$ <br>  $= \int_0^t A\mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds$ <br>  $+ \int_0^t A\mathcal{P}_{\alpha}(t-s)g(t)ds$ <br>
From Lemma 3.2 and (H), we<br>  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) -$ <br>  $+ \int_0^t A\mathcal{P}_{\alpha}(t-s)g(t)ds$ <br>  $+ \mathcal{P}_{\alpha}(t)g(t) \$  $\begin{array}{ll} \mbox{and $A\mathcal{P}_{\alpha}(t)u_0$ are Lipschit}\\ \mbox{From Lemma 3.2 and (H)},\\ \mbox{From Lemma 3.2 and (H)},\\ \mbox{From Lemma 3.2 and (H)},\\ \mbox{For $a$ fixed $t$ $\in$ $(0,T]$, from Lemma 2.4 and}\\ \mbox{(H), these two integrals converge absolutely. There}\\ \mbox{for $ \varphi(t) = \int_0^t \mathcal{P}'_\alpha(t-s)g(t)ds$} & \mbox{This follows from similar}\\ \mbox{(H), these two integrals converge absolutely. There}\\ \mbox{for $ \varphi(t) \in D(A)$. By repeating the reasoning used in the proof of Lemma 3.2, we have $A\Phi$  $\begin{array}{ll} A\Phi(t)=A\Phi_1(t)+A\Phi_2(t) & \mbox{From Lemma 3.2 and (F)}\\ \hline \\ =\int_0^t A\mathcal{P}_{\alpha}(t-s)(g(s)-g(t))ds & \Phi'(t)=\int_0^t \mathcal{P}'_{\alpha}(t-s)(g(t))ds \\ &+\int_0^t A\mathcal{P}_{\alpha}(t-s)g(t)ds & +\mathcal{P}_{\alpha}(t) \end{array}$  <br> For a fixed  $t\in(0,T],$  from Lemma 2.4 and <br> (H), these two integrals con  $C((0,T];L^2(\Omega))$ . To prove a similar =  $\int_0^t A \mathcal{P}_{\alpha}(t-s)(g(s) - g(t))ds$ <br>+  $\int_0^t A \mathcal{P}_{\alpha}(t-s)g(t)ds$ <br>a fixed  $t \in (0, T]$ , from Lemma 2.4<br>, these two integrals converge absolutely. The<br>,  $\Phi(t) \in D(A)$ . By repeating the reaso<br>d in the proof of Lemma 3.2, we have Aq<br> $\label{eq:1.1} \begin{aligned} &\int_0^t A \mathcal{P}_{\alpha}(t-s) (g(s)-g(t)) ds &\qquad \Phi'(t)=\int_0^t \mathcal{P}'_{\alpha}(t-s) (g(s)-g(t)) ds \\ &+\int_0^t A \mathcal{P}_{\alpha}(t-s) g(t) ds &+\mathcal{P}_{\alpha}(t) g(t) \in C^\theta \\ \in &\quad (0,T], \text{ from Lemma 2.4 and}\\ &\text{thegrals converge absolutely. There-}\\ &\text{(A). By repeating the reasoning}\\ &\text{for of Lemma 3.2, we have } A\Phi_1\in &\quad \text{therefore, }\partial_t^\alpha A u = g(t)-\partial_t u - Au \in C^\$  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds$ <br>  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds$ <br>  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds$ <br>
For a fixed  $t \in (0, T]$ , from Lemma 2.4 and<br>
(H), these two integrals converge absolutely. There-<br>
fore,  $\Phi(t) \$  $+\int_0^t A\mathcal{P}_{\alpha}(t-s)g(t)ds$ <br>For a fixed  $t \in (0,T]$ , from Lemma 2.4<br>(H), these two integrals converge absolutely. T<br>fore,  $\Phi(t) \in D(A)$ . By repeating the reas<br>used in the proof of Lemma 3.2, we have A<br> $C((0,T]; L^2(\Omega))$ . To prove a

$$
A\Phi_2(t+h) - A\Phi_2(t)
$$
\nThe R  
\n
$$
= \int_0^h A\mathcal{P}_{\alpha}(t+h-s)g(t+h)ds
$$
\nThe Newton  
\ntion due  
\n
$$
+ \int_h^{t+h} A\mathcal{P}_{\alpha}(t+h-s)g(t+h)ds
$$
\nflow pro  
\ncausal  
\n
$$
- \int_0^t A\mathcal{P}_{\alpha}(t-s)g(t)ds
$$
\nThe ma  
\ning the  
\n
$$
+ \int_0^h A\mathcal{P}_{\alpha}(t+h-s)g(t+h)ds
$$
\nin this I  
\n
$$
+ \int_0^t A\mathcal{P}_{\alpha}(t-s)(g(t+h) - g(t))ds
$$
\nIn this I  
\nproblem

*Nguyen Nhu Quan/Vol 10.* No 4\_August 2024 | p.26-32  
\nas 
$$
h \to 0
$$
. From Lemma 2.2(i), we deduce that  
\n
$$
\lim_{h\to 0} \frac{1}{h} \int_0^h P_{\alpha}(s)g(t) ds = g(t).
$$
\n
$$
\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} P_{\alpha}(t+h-s)g(s) ds = g(t).
$$
\n
$$
\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} P_{\alpha}(t+h-s)g(s) ds = g(t).
$$
\n
$$
\leq C ||g(t+h)|| \int_0^h (t+h-s)^{\alpha-1} ds
$$
\nThis means that  $\Phi(t)$  is continuously differentiable  
\nat  $t$  and its derivative, denoted  $\Phi_+(t)$ , satisfies:  
\n
$$
\Phi'_+(t) = \int_0^t P'_{\alpha}(t-s)g(s) ds + g(t).
$$
\nThis means that  $AB_2 \in C$   $(\epsilon, T]$ ;  $L^2(\Omega)$ ). There-  
\nBy reasoning as above, assuming that  $0 < h < t$ ,  
\nwe can easily deduce that  $\Phi(t)$  is differentiable  
\n $t \in (0, T]$ ;  $L^2(\Omega)$  because  $\epsilon$  is arbitrary.  
\nWe obtain:  
\n $t = \text{and } \Phi'_+(t) = \Phi'_-(t)$ . From Lemma 3.2 and (H),  
\nWe obtain:  
\n
$$
- (1 + k\partial_t^{\alpha}) A \Phi(t) = -\frac{d}{dt} (\eta * A \Phi(t))
$$
\n
$$
= \frac{d}{dt} ((\mathcal{P}_{\alpha}(t) - I) * g)
$$

 $([\epsilon, T]; L^2(\Omega))$ . Therefore,  $A\Phi_2 \in C((0, T]; L^2(\Omega))$  because  $\epsilon$  is arbitr  $\alpha^{\alpha} A \Phi \in C\left((0,T];L^2(\Omega)\right).$  $J_0$ <br>  $||g(s)||\epsilon^{\alpha-1}h + CL_1h^{\theta}\frac{T^{\alpha}}{\alpha}$ . (3.5)<br>
it  $A\Phi_2 \in C([{\epsilon, T}]; L^2(\Omega))$ . There-<br>  $(0, T]; L^2(\Omega)$  because  ${\epsilon}$  is arbitrary.<br>
it that  $\partial_t^{\alpha} A\Phi \in C((0, T]; L^2(\Omega))$ .<br>
2(ii), we have:<br>  $A\Phi(t) = -\frac{d}{dt}(\eta * A\Phi(t))$ <br>  $= \frac{d}{dt}((\mathcal$ This means that  $A\Phi_2 \in C([e, T]; L^2(\Omega))$ . There-<br>fore,  $A\Phi_2 \in C((0, T]; L^2(\Omega))$  because  $\epsilon$  is arbitrary.<br>Next, we prove that  $\partial_t^{\alpha} A \Phi \in C((0, T]; L^2(\Omega))$ .<br>From Lemma 2.2(ii), we have:<br> $-(1 + k\partial_t^{\alpha}) A\Phi(t) = -\frac{d}{dt} (\eta * A\Phi(t))$ <br> $= \frac{d}{$ 

$$
-(1 + k\partial_t^{\alpha}) A\Phi(t) = -\frac{d}{dt} (\eta * A\Phi(t))
$$
  
= 
$$
\frac{d}{dt} ((\mathcal{P}_{\alpha}(t) - I) * g)
$$
  
= 
$$
\Phi'(t) - g(t).
$$

 $\begin{array}{rclcrcl} \Delta' (t-s)g(t)ds + g(t) & A\Phi & = & A\Phi_1 & + & A\Phi_2 & \in & C\left((0,T];L^2(\Omega)\right) \end{array}$ mma 3.2 and (H),<br>  $-(1 + k\partial_t^{\alpha}) A\Phi(t)$ <br>  $y(t)$ <br>  $\qquad$   $\$  $h^2(Q)$  has been proven, we conclude that  $\partial_t^{\alpha} A \Phi \in$ Next, we prove that  $\partial_t^{\alpha} A \Phi \in C((0, T]; L^2(\Omega)).$ <br>
From Lemma 2.2(ii), we have:<br>  $-(1 + k\partial_t^{\alpha}) A \Phi(t) = -\frac{d}{dt} (\eta * A \Phi(t))$ <br>  $= \frac{d}{dt} ((\mathcal{P}_{\alpha}(t) - I) * g)$ <br>  $= \Phi'(t) - g(t).$ <br>
Here, \* denotes the convolution operator. Since<br>  $A\Phi = A\Phi_1 + A\$  $[0, T]; L<sup>2</sup>(\Omega)).$ <br>  $[D(t))$ <br>  $(I) * g$ <br>  $(L)$ <br>  $\Gamma(t) * g$ From Lemma 2.2(ii), we have:<br>  $-(1 + k\partial_t^{\alpha}) A\Phi(t) = -\frac{d}{dt} (\eta * A\Phi(t))$ <br>  $= \frac{d}{dt} ((\mathcal{P}_{\alpha}(t) - I) * g)$ <br>  $= \Phi'(t) - g(t).$ <br>
Here, \* denotes the convolution operator. Since<br>  $A\Phi = A\Phi_1 + A\Phi_2 \in C((0, T]; L^2(\Omega))$ <br>
has been proven, we conclude  $C((0, T]; L^2(\Omega))$ . Moreover,  $u(t)$  =  $-(1 + k\partial_t^{\alpha}) A\Phi(t) = -\frac{d}{dt} (\eta * A\Phi(t))$ <br>  $= \frac{d}{dt} ((\mathcal{P}_{\alpha}(t) - I) * g)$ <br>  $= \Phi'(t) - g(t).$ <br>
re, \* denotes the convolution operator. S<br>  $= A\Phi_1 + A\Phi_2 \in C((0, T]; L^2)$ <br>
been proven, we conclude that  $\partial_t^{\alpha} A\Phi$ <br>
(0, T]; L<sup>2</sup>(Ω)). Moreove  $A\Phi(t) = -\frac{d}{dt}(\eta * A\Phi(t))$ <br>  $= \frac{d}{dt}((\mathcal{P}_{\alpha}(t) - I) * g)$ <br>  $= \Phi'(t) - g(t).$ <br>
the convolution operator. Since<br>  $+ A\Phi_2 \in C((0, T]; L^2(\Omega))$ <br>
n, we conclude that  $\partial_t^{\alpha} A\Phi \in$ . Moreover,  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \Phi(t)$ <br>
tion of (3.1).<br>
mma 2  $-(1 + k\partial_t^{\alpha}) A\Phi(t) = -\frac{d}{dt} (\eta * A\Phi(t))$ <br>  $= \frac{d}{dt} ((\mathcal{P}_{\alpha}(t) - I) * g)$ <br>  $= \Phi'(t) - g(t).$ <br>Here, \* denotes the convolution operator. Since<br> $A\Phi = A\Phi_1 + A\Phi_2 \in C((0, T]; L^2(\Omega))$ <br>has been proven, we conclude that  $\partial_t^{\alpha} A\Phi \in C((0, T]; L^2(\Omega))$ .  $\begin{aligned}\n&= \frac{d}{dt} \left( (\mathcal{P}_{\alpha}(t) - I) * g \right) \\
&= \Phi'(t) - g(t).\n\end{aligned}$ <br>
Here, \* denotes the convolution operator. Since<br>  $A\Phi = A\Phi_1 + A\Phi_2 \in C \left( (0, T]; L^2(\Omega) \right)$ <br>
has been proven, we conclude that  $\partial_t^{\alpha} A\Phi \in C \left( (0, T]; L^2(\Omega) \right)$ . More  $=\frac{1}{dt}((P_{\alpha}(t)-1)*g)$ <br>  $=\Phi'(t)-g(t).$ <br>
Here, \* denotes the convolution operator. Since<br>  $A\Phi = A\Phi_1 + A\Phi_2 \in C((0,T]; L^2(\Omega))$ <br>
has been proven, we conclude that  $\partial_t^{\alpha} A\Phi \in C((0,T]; L^2(\Omega))$ . Moreover,  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \Phi(t)$ <br>
is a cl =  $A\Phi_1 + A\Phi_2 \in C((0, T]; L^2(\Omega))$ <br>been proven, we conclude that  $\partial_t^{\alpha} A\Phi$ <br> $(0, T]; L^2(\Omega))$ . Moreover,  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \Phi$ <br>classical solution of (3.1).<br>ording to Lemma 2.3, we obtain that  $\mathcal{P}'_{\alpha}(t)$ <br> $AP_{\alpha}(t)u_0$  are Li

According to Lemma 2.3, we obtain that  $\mathcal{P}'_{\alpha}(t)u_0$ 

has been proven, we conclude that 
$$
\partial_t^T A \Phi \in
$$
  
\n $C((0, T]; L^2(\Omega))$ . Moreover,  $u(t) = \mathcal{P}_{\alpha}(t)u_0 + \Phi(t)$   
\nis a classical solution of (3.1).  
\nAccording to Lemma 2.3, we obtain that  $\mathcal{P}'_{\alpha}(t)u_0$   
\nand  $A\mathcal{P}_{\alpha}(t)u_0$  are Lipschitz continuous on  $(0, T]$ .  
\nFrom Lemma 3.2 and (H), we have:  
\n
$$
\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t - s)(g(s) - g(t))ds
$$
\n
$$
+ \mathcal{P}_{\alpha}(t)g(t) \in C^{\theta}((0, T]; L^2(\Omega))
$$
\nThis follows from similar reasoning used in the proof of Lemma 3.2 and equation (3.5), which  
\nshows that  $A\Phi = A\Phi_1 + A\Phi_2 \in C^{\theta}((0, T]; L^2(\Omega))$ .

s a classical solution of (3.1).<br>
According to Lemma 2.3, we obtain that  $\mathcal{P}'_{\alpha}(t)u_0$ <br>
and  $A\mathcal{P}_{\alpha}(t)u_0$  are Lipschitz continuous on  $(0, T]$ .<br>
From Lemma 3.2 and (H), we have:<br>  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t - s)(g(s) - g(t))ds$ According to Lemma 2.3, we obtain that  $\mathcal{P}'_{\alpha}(t)u_0$ <br>and  $A\mathcal{P}_{\alpha}(t)u_0$  are Lipschitz continuous on  $(0, T]$ .<br>From Lemma 3.2 and (H), we have:<br> $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds$ <br> $+ \mathcal{P}_{\alpha}(t)g(t) \in C^{\theta}((0, T]; L^2(\Omega))$ <br>  $\theta$   $((0,T];L^2(\Omega)).$ Notice in  $(0, T]$ .<br>  $s$ <br>  $(0, T]; L^2(\Omega)$ <br>  $(3.5)$ , which<br>  $(0, T]; L^2(\Omega)$ .<br>  $(L^2(\Omega))$ , and<br>  $(L^2(\Omega))$ . This . From Lemma 3.2 and (H), we have:<br>  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds$ <br>  $+ \mathcal{P}_{\alpha}(t)g(t) \in C^{\theta}((0,T]; L^2(\Omega))$ <br>
This follows from similar reasoning used in the<br>
proof of Lemma 3.2 and equation (3.5), which<br>
shows that  $A\Phi = A\Phi$  $\theta$   $((0,T];L^2(\Omega)), \text{ and}$ have:<br>  $g(t)$ )ds<br>  $C^{\theta} ((0, T]; L^2(\Omega))$ <br>
soning used in the<br>
uation (3.5), which<br>  $\in C^{\theta} ((0, T]; L^2(\Omega))$ .<br>  $(0, T]; L^2(\Omega))$ , and<br>  $((0, T]; L^2(\Omega))$ . This  $(0, T].$ <br>
( $\Omega$ )<br>
in the<br>
which<br>  $(2^{2}(\Omega)).$ <br>
, and<br>
. This  $k\partial_t^{\alpha} Au = g(t) - \partial_t u - Au \in C^{\theta} ((0, T]; L^2(\Omega))$ . This  $C^{\theta} ((0, T]; L^2(\Omega))$ <br>soning used in the<br>ation (3.5), which<br> $C^{\theta} ((0, T]; L^2(\Omega)).$ <br> $(0, T]; L^2(\Omega)),$  and<br> $(0, T]; L^2(\Omega)).$  This  $\Omega$ )<br>
n the<br>
which<br>  $\binom{2}{\Omega}$ ).<br>
and<br>
This  $\Phi'(t) = \int_0^t \mathcal{P}'_{\alpha}(t-s)(g(s) - g(t))ds$ <br>  $+ \mathcal{P}_{\alpha}(t)g(t) \in C^{\theta}((0,T]; L^2(\Omega))$ <br>This follows from similar reasoning used in the<br>proof of Lemma 3.2 and equation (3.5), which<br>shows that  $A\Phi = A\Phi_1 + A\Phi_2 \in C^{\theta}((0,T]; L^2(\Omega))$ .<br>Therefore  $+ \mathcal{P}_{\alpha}(t)g(t) \in C^{\theta}((0, T]; L^{2}(\Omega))$ <br>This follows from similar reasoning used in<br>proof of Lemma 3.2 and equation (3.5), wh<br>shows that  $A\Phi = A\Phi_{1} + A\Phi_{2} \in C^{\theta}((0, T]; L^{2}(\Omega))$ .<br>Therefore,  $\partial_{t}u, Au \in C^{\theta}((0, T]; L^{2}(\Omega))$ ,  $k\partial_{$ This follows from similar reasoning used in the<br>proof of Lemma 3.2 and equation (3.5), which<br>shows that  $A\Phi = A\Phi_1 + A\Phi_2 \in C^{\theta}((0,T]; L^2(\Omega))$ .<br>Therefore,  $\partial_t u, Au \in C^{\theta}((0,T]; L^2(\Omega))$ , and<br> $k\partial_t^{\alpha} Au = g(t) - \partial_t u - Au \in C^{\theta}((0,T]; L^2(\Omega))$ proof of Lemma 3.2 and equation (3.5), which<br>shows that  $A\Phi = A\Phi_1 + A\Phi_2 \in C^{\theta}((0,T]; L^2(\Omega)).$ <br>Therefore,  $\partial_t u, Au \in C^{\theta}((0,T]; L^2(\Omega)),$  and<br> $k\partial_t^{\alpha} Au = g(t) - \partial_t u - Au \in C^{\theta}((0,T]; L^2(\Omega)).$  This<br>completes the proof. <br>4 CONCLUSION<br>The Rayle

flow problem examines the motion of a fluid flow shows that  $A\Phi = A\Phi_1 + A\Phi_2 \in C^{\theta}((0,T]; L^2(\Omega)).$ <br>
Therefore,  $\partial_t u, Au \in C^{\theta}((0,T]; L^2(\Omega)),$  and<br>  $k\partial_t^{\alpha}Au = g(t) - \partial_t u - Au \in C^{\theta}((0,T]; L^2(\Omega)).$  This<br>
completes the proof. <br>
4 CONCLUSION<br>
The Rayleigh-Stokes problem for certain non-<br>
New Therefore,  $\partial_t u, Au \in C^{\theta} ((0,T]; L^2(\Omega))$ , and<br>  $k\partial_t^{\alpha} Au = g(t) - \partial_t u - Au \in C^{\theta} ((0,T]; L^2(\Omega))$ . This<br>
completes the proof.  $\square$ <br>
4 CONCLUSION<br>
The Rayleigh-Stokes problem for certain non-<br>
Newtonian fluids has received considerable a  $k\partial_t^{\alpha} Au = g(t) - \partial_t u - Au \in C^{\theta} ((0, T]; L^2(\Omega)).$  This<br>completes the proof.  $\square$ <br>4 CONCLUSION<br>The Rayleigh-Stokes problem for certain non-<br>Newtonian fluids has received considerable atten-<br>tion due to its practical importance. Thi completes the proof.  $\Box$ <br>4 CONCLUSION<br>The Rayleigh-Stokes problem for certain non-<br>Newtonian fluids has received considerable atten-<br>tion due to its practical importance. This unsteady<br>flow problem examines the motion of 4 CONCLUSION<br>The Rayleigh-Stokes problem for certain non-<br>Newtonian fluids has received considerable atten-<br>tion due to its practical importance. This unsteady<br>flow problem examines the motion of a fluid flow<br>caused by a s 4 CONCLUSION<br>The Rayleigh-Stokes problem for certain non-<br>Newtonian fluids has received considerable atten-<br>tion due to its practical importance. This unsteady<br>flow problem examines the motion of a fluid flow<br>caused by a s The Rayleigh-Stokes problem for certain non-<br>Newtonian fluids has received considerable atten-<br>tion due to its practical importance. This unsteady<br>flow problem examines the motion of a fluid flow<br>caused by a sudden displac The Rayleigh-Stokes problem for certain non-<br>Newtonian fluids has received considerable atten-<br>tion due to its practical importance. This unsteady<br>flow problem examines the motion of a fluid flow<br>caused by a sudden displac

problem for generalized second-order fluids. Our

Nguyen Nhu Quan/Vol 10. No 4\_August 2024| p.26-32<br>goal is to establish the fundamental theory of so-<br>lutions to this equation. In particular, the exis-<br>tence and regularity of classical solutions are in-<br>lem for a maxwell *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>goal is to establish the fundamental theory of so-<br>lutions to this equation. In particular, the exis-<br>tence and regularity of classical solutions are in-<br>vestigated. The *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>goal is to establish the fundamental theory of so-<br>lutions to this equation. In particular, the exis-<br>Fetecau, Z. J., C. (2003). The rayle<br>tence and regularity of classi vestigated is to establish the fundamental theory of so-<br>
vestigated. The proof of the main results relies on the main results relies on the fixed-point technique and estimates for the re-<br>
vestigated. The proof of the ma *Nguyen Nhu Quan/*Vol 10. No 4\_August 2024| p.26-32<br>goal is to establish the fundamental theory of so-<br>lutions to this equation. In particular, the exis-<br>fetecau, Z. J., C. (2003). The ray<br>tence and regularity of classica Nguyen Nhu Quan/Vo<br>goal is to establish the fundamental theory of<br>lutions to this equation. In particular, the ex-<br>tence and regularity of classical solutions are<br>vestigated. The proof of the main results relies<br>the fixedgoal is to establish the fundamental theory of socheral indications to this equation. In particular, the exisered and z. J., C. (2003). The tence and regularity of classical solutions are in-<br>
vestigated. The proof of the s to this equation. In particular, the exis-<br>
and regularity of classical solutions are in-<br>
and regularity of classical solutions are in-<br>
and for a maxwell fluin ated. The proof of the main results relies on<br>  $Phys.$  54, and regularity of classical solutions are in-<br>
in for a maxwell ated. The proof of the main results relies on  $Phys$ ,  $54$ ,  $1086-109$ ;<br>
red-point technique and estimates for the re-<br>
Kilbas, S. H. T. J., A.A.<br>
toperator. <br>

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