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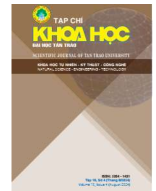

NONLOCAL RAYLEIGH-STOKES FLOWS: EXISTENCE AND REGULARITY RESULTS

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Abstract:

We investigate the Rayleigh-Stokes problem for a generalized second-grade fluid with a fractional time derivative. Our study aims to prove the existence and regularity of classical solutions using fixed point techniques and asymptotic estimates of the resolvent operator.

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DÒNG CHẢY RAYLEIGH-STOKES KHÔNG ĐỊA PHƯƠNG: KẾT QUẢ VỀ SỰ TỒN TẠI VÀ TÍNH CHÍNH QUY

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Sự tồn tại nghiệm, Tính chính quy, Nghiệm tích phân, Bài toán Rayleigh–Stokes, Đạo hàm Riemann–Liouville.

Tóm tắt:

Chúng tôi nghiên cứu bài toán với phương trình Rayleigh-Stokes bậc hai tổng quát cho chuyển động của chất lỏng với đạo hàm phân số theo thời gian. Nghiên cứu tập trung vào sự tồn tại và tính chính quy của nghiệm cổ điển, sử dụng kỹ thuật điểm bất động và các ước lượng tiệm cận của toán tử giải thức.

1 INTRODUCTION

The article studies the following general Cauchy problem

$$\begin{cases} \partial_t u = -\left(1 + k\partial_t^\alpha\right) Au + g(t), t > 0 \\ u(0) = u_0 \end{cases} \tag{1.1}$$

arises from the Rayleigh-Stokes problem for the following general second-order fluid model:

$$\begin{cases} \partial_t u - \left(1 + k\partial_t^\alpha\right) \Delta u = f(t,u), & x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega \end{cases} \tag{1.2}$$

with $t \in (0,T)$, where $k > 0$ is a parameter, Ω is a bounded domain of $\mathbb{R}^d$ with a smooth boundary, $T \in (0,\infty)$, and ∂_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined by (Kilbas, 2006; Y. Zhou, 2014):

$$\partial_t^\alpha u(t,x) = \frac{\partial}{\partial t}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} u(s,x)ds\right) \tag{1.3}$$

with the right-hand side defined at each point in $(0,T)$. Here, $\Gamma(\cdot)$ is the Gamma function, that is, $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$. Let $A = -\Delta$ in $L^2(\Omega)$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then, we rewrite (1.2) in the form (1.1), where u_0 is the initial value in $L^2(\Omega)$ and $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ is a given function satisfying certain assumptions.

Recently, the Rayleigh-Stokes problem for some non-Newtonian fluids, such as Oldroyd-B fluids, Maxwell fluids, and second-order fluids, has attracted significant attention due to their physical importance (Fetecau, 2003). This unstable flow problem considers the motion of the fluid flow caused by a sudden displacement of the boundary. The mathematical model is derived by combining the laws of conservation in physics with the constitutive relationship of second-order viscoelastic fluids.

$$\rho\left(\frac{d\mathbf{V}}{dt} - (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = \nabla\cdot(-p\mathbf{I} + \mu_0\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2)$$

where ρ is the density of the fluid, $\mathbf{V}$ is the velocity, p is the hydrostatic pressure, $\mu_0 \geq 0$, $\alpha_1 \geq 0$, and $\alpha_1 + \alpha_2 = 0$. $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined by:

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T,$$
$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_1.$$

2 PRELIMINARIES

We introduce some notation. For any Banach space X, we denote $B(X)$ as the space of all bounded linear operators on X. The special beta function $\mathcal{B} : (0,\infty)\times(0,\infty)\to(0,\infty)$ is defined by:

$$\mathcal{B}(a,b) = \int_0^1 (1-s)^{a-1}s^{b-1}ds$$

We have

$$\mathcal{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

In the rest of this section, we first introduce a family of operators and then use representation formulas to prove the main estimates for this family of operators, which will be used throughout the paper.

Using the results from (Bazhlekova, 2015; W. J. Zhou Y., 2021), we introduce a family of operators $\{\mathcal{P}_\alpha(t)\}_{t\geq 0}$ defined by

$$\mathcal{P}_\alpha(t)v = \sum_{n=1}^{\infty} \mathcal{S}_n(\alpha,t)\,(v,\varphi_n)\,\varphi_n, \tag{2.1}$$

with $v \in L^2(\Omega)$, where

$$\mathcal{S}_n(\alpha,t) = \frac{1}{2\pi i}\int_{Br} e^{zt}\frac{1}{z + k\lambda_n z^\alpha + \lambda_n}dz = \int_0^\infty e^{-rt}\mathcal{K}_n(r)dr \tag{2.2}$$

where $B_r = \{z : \operatorname{Re} z = \sigma, \sigma > 0\}$ is the Bromwich contour and

$$\mathcal{K}_n(r) = \frac{k}{\pi}\frac{\lambda_n r^\alpha \sin\alpha\pi}{\psi^2 + (\lambda_n k r^\alpha \sin\alpha\pi)^2}. \tag{2.3}$$

with $\psi = (-r + \lambda_n k r^\alpha\cos(\alpha\pi) + \lambda_n)$. Since $\alpha \in (0,1)$ and $\lambda_n, k > 0$, we have $\mathcal{K}_n(r) > 0$ for all $r > 0$.

Lemma 2.1. *The functions $\mathcal{S}_n(\alpha,t)$, $n = 1,2,\ldots$, have the following properties:*

(i) $\mathcal{S}_n(\alpha,0) = 1$;

(ii) There exists a constant $C = C(k,\alpha) > 0$ such that

$$\mathcal{S}_n(\alpha,t) \leq \frac{C}{1+\lambda_n t^{1-\alpha}}.$$

Proof. Property (i) has been proven in (Bazhlekova, 2015, Theorem 2.2), and Property (ii) has been proven in (W. J. Zhou Y., 2021, Lemma 3.1). □

From Lemma 2.1(ii), it can be seen that $\mathcal{P}_\alpha(t)$ is bounded in $L^2(\Omega)$ for all $t \geq 0$.

Lemma 2.2. *Let $\{\mathcal{P}_\alpha(t)\}_{t\geq 0}$ be a family of linear operators defined by (2.1). Then:*

(i) $\mathcal{P}_\alpha(t)$ is strongly continuous on $\mathbb{R}_+$. Furthermore, for all $\delta > 0$, the continuity is uniform on $[\delta,\infty)$;

(ii) For every $v \in D(A)$ and $t \geq 0$,

$$\mathcal{P}_\alpha(t)v = v - \int_0^t \eta(t-s)A\mathcal{P}_\alpha(s)v\,ds,$$

where $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)}t^{-\alpha}$.

Lemma 2.3. *For every $\delta > 0$ and $v \in L^2(\Omega)$, $\mathcal{P}'_\alpha(t)v$ and $A\mathcal{P}_\alpha(t)v$ are Lipschitz continuous on $\delta \leq t \leq T$. Moreover, for $0 < s < t \leq T$, we have the following estimates:*

$$\|\mathcal{P}'_\alpha(t) - \mathcal{P}'_\alpha(s)\|_{B(L^2(\Omega))} \leq \frac{t-s}{st},$$

and

$$\|A\mathcal{P}_\alpha(t) - A\mathcal{P}_\alpha(s)\|_{B(L^2(\Omega))} \leq C_1 T^\alpha\frac{t-s}{st},$$

where C_1 is a constant dependent on the problem parameters.

Lemma 2.4. *For $0 < \gamma \leq 1$, for the family $\{\mathcal{P}_\alpha(t)\}_{t\geq 0}$ defined by (2.1), we have the following results:*

$$\|A^\gamma\mathcal{P}_\alpha(t)\|_{B(L^2(\Omega))} \leq Ct^{(\alpha-1)\gamma}, \quad t > 0,$$

where C is a constant depending on the problem parameters.

Moreover, for every $v \in L^2(\Omega)$,

$$\lim_{t\to 0} t^{(1-\alpha)\gamma} \|A^\gamma \mathcal{P}_\alpha(t) v\| = 0.$$

3 MAIN RESULTS

To provide a suitable definition for the solutions of (1.1), we study the following linear problem:

$$\begin{cases} \partial_t u + (1 + k\partial_t^\alpha) Au = g(t), t > 0 \\ u(0) = u_0 \end{cases} \tag{3.1}$$

By integrating both sides of the equation (3.1), we obtain:

$$u(t) = u_0 - \int_0^t \eta(t-s) Au(s) ds + \int_0^t g(s) ds \tag{3.2}$$

Note that $\eta(t) = 1 + \frac{k}{\Gamma(1-\alpha)} t^{-\alpha}$. Before presenting the definition of the integral solution to (3.1), we state the following lemma.

Lemma 3.1. *If*

$$u(t) = u_0 - \int_0^t \eta(t-s) Au(s) ds + \int_0^t g(s) ds, t \in [0, T]$$

then

$$u(t) = \mathcal{P}_\alpha(t) u_0 + \int_0^t \mathcal{P}_\alpha(t-s) g(s) ds. \tag{3.3}$$

Proof. Using the Laplace transform for the equation (3.2), we obtain:

$$\widehat{u}(\lambda) = \lambda^{-1} u_0 - \widehat{\eta}(\lambda) A \widehat{u}(\lambda) + \lambda^{-1} \widehat{g}(\lambda).$$

It means

$$(I + \widehat{\eta}(\lambda) A) \widehat{u}(\lambda) = \lambda^{-1} u_0 + \lambda^{-1} \widehat{g}(\lambda)$$

We have

$$\begin{aligned} &\widehat{u}(\lambda) = (I + \widehat{\eta}(\lambda) A)^{-1} \left(\lambda^{-1} u_0 + \lambda^{-1} \widehat{g}(\lambda)\right) \\ &= \int_0^\infty e^{-\lambda t} \left(\mathcal{P}_\alpha(t) u_0 + \int_0^t \mathcal{P}_\alpha(t-s) g(s) ds\right) dt \end{aligned}$$

the proof is complete. □

From the above discussions, we use the following concept of an integral solution for the problem (3.1).

Definition 3.1. *The function $u : [0, T] \to L^2(\Omega)$ is called an integral solution of (3.1) if $u \in C\left([0, T], L^2(\Omega)\right)$ and satisfies (3.3).*

Assume that $u_0 \in L^2(\Omega)$ and $g \in L^1\left(0, T, L^2(\Omega)\right)$. Lemma 3.1 shows that the problem (3.1) has a unique integral solution. We are interested in imposing additional conditions on g so that the integral solution becomes a classical solution.

Definition 3.2. *The function $u : [0, T] \to L^2(\Omega)$ is called a classical solution of (3.1) if $u \in C\left([0, T]; L^2(\Omega)\right)$ with $\partial_t u \in C\left((0, T]; L^2(\Omega)\right)$, $u(t) \in D(A)$ for all $t \in (0, T]$, $\partial_t^\alpha Au \in C\left((0, T]; L^2(\Omega)\right)$, and satisfies (3.1).*

In this paper, we assume the following:

(H) The function $g(t)$ is Hölder continuous with exponent $\theta \in (0, 1)$, meaning that

$$\|g(t) - g(s)\| \le L_1 |t-s|^\theta, \quad \text{for all } 0 < s, t \le T,$$

where L_1 is a constant.

Lemma 3.2. *Assume that the assumption (H) is satisfied. If*

$$w(t) = \int_0^t \mathcal{P}'_\alpha(t-s)(g(s) - g(t))\, ds, \quad t \in (0, T],$$

then $w(t) \in C^\theta\left((0, T]; L^2(\Omega)\right)$.

Proof. For $0 < t < t + h \le T$, we have:

$$\begin{aligned} &w(t+h) - w(t) \\ &= \int_0^t \left(\mathcal{P}'_\alpha(t+h-s) - \mathcal{P}'_\alpha(t-s)\right)(g(s) - g(t)) ds \\ &\quad + \int_0^t \mathcal{P}'_\alpha(t+h-s)(g(t) - g(t+h)) ds \\ &\quad + \int_t^{t+h} \mathcal{P}'_\alpha(t+h-s)(g(s) - g(t+h)) ds \\ &= I_1 + I_2 + I_3 \end{aligned}$$

We estimate each component in the three components separately. For I_1, based on

$$\begin{aligned} \mathcal{P}'_\alpha(t) v = \sum_{n=1}^\infty \mathcal{S}'_n(\alpha, t) (v, \varphi_n) \varphi_n, &\|\mathcal{P}'_\alpha(t)\|_{B(L^2(\Omega))} \\ \le \frac{1}{t}. \end{aligned} \tag{3.4}$$

With $t > 0$ and (H), we have:

$$\begin{aligned} &\|\mathcal{P}'_\alpha\ (t+h-s)(g(s) - g(t))\| \\ &\quad \le L_1 (t+h-s)^{-1} (t-s)^\theta \\ &\quad \le L_1 (t-s)^{\theta - 1} \end{aligned}$$

for all $0 < h \le T - t$, the Dominated Convergence Theorem (Lebesgue's Dominated Convergence Theorem) implies that $I_1 \to 0$ as $h \to 0$.

Furthermore, using Lemma 2.3, we obtain:

$$\begin{aligned}\|I_1\| &\le \int_0^t \|(\mathcal{P}'_\alpha(t+h-s)\\ &\qquad -\mathcal{P}'_\alpha(t-s))\,(g(s)-g(t))\|\,ds\\ &\le h\int_0^t \frac{1}{(t+h-s)(t-s)}\|(g(s)-g(t))\|ds\\ &\le L_1 h\int_0^t \frac{s^{\theta-1}}{s+h}ds\\ &\le L_1\int_0^h \frac{h}{s+h}s^{\theta-1}ds + L_1 h\int_h^\infty \frac{s}{s+h}s^{\theta-2}ds\\ &\le L_1\frac{h^\theta}{\theta(1-\theta)}.\end{aligned}$$

Similarly, I_3 is estimated as follows:

$$\begin{aligned}\|I_3\| &\le \int_t^{t+h}(t+h-s)^{-1}\|(g(s)-g(t+h))\|ds\\ &\le L_1\int_t^{t+h}(t+h-s)^{\theta-1}ds\\ &\le L_1\frac{h^\theta}{\theta}\end{aligned}$$

The boundedness of $\mathcal{P}_\alpha(t)$ implies that:

$$\begin{aligned}\|I_2\| &= \|(\mathcal{P}_\alpha(t+h)-\mathcal{P}_\alpha(h))\,(g(t)-g(t+h))\|\\ &\le 2CL_1h^\theta\end{aligned}$$

By combining the estimates for I_1, I_2, and I_3, we obtain the desired result. □

Now we prove the main result of this paper in the following theorem:

Theorem 3.3. *Let $u_0 \in L^2(\Omega)$ and (H) be satisfied. Then, the integral solution of* (3.1) *is a classical solution. Moreover, we have $\partial_t u, \partial_t^\alpha Au \in C^\theta\left((0,T];L^2(\Omega)\right)$.*

Proof. Let u be the integral solution of the problem (3.1). Note that

$$u(t) = \mathcal{P}_\alpha(t)u_0 + \int_0^t \mathcal{P}_\alpha(t-s)g(s)ds, t\in[0,T]$$

With $u_0 \in L^2(\Omega)$, Lemma 2.2(ii) implies that $\mathcal{P}_\alpha(t)u_0$, for $t>0$, is the classical solution of the following problem:

$$\begin{cases}\partial_t u = -\left(1+k\partial_t^\alpha\right)Au, t>0\\ u(0)=u_0\end{cases}$$

Let us recall that

$$\Phi(t) = \int_0^t \mathcal{P}_\alpha(t-s)g(s)ds$$

is a classical solution of the following problem:

$$\begin{cases}\partial_t u + \left(1+k\partial_t^\alpha\right)Au = g(t), t>0\\ u(0)=0\end{cases}$$

For a fixed $t\in(0,T]$, we prove that $\Phi(t)$ is continuously differentiable at t. Let $0<h\le T-t$, then:

$$\begin{aligned}\frac{\Phi(t+h)-\Phi(t)}{h} &= \frac{1}{h}\left(\int_0^{t+h}\mathcal{P}_\alpha(t+h-s)g(s)ds\right.\\ &\qquad \left.-\int_0^t \mathcal{P}_\alpha(t-s)g(s)ds\right)\\ &= \int_0^t \frac{\mathcal{P}_\alpha(t+h-s)-\mathcal{P}_\alpha(t-s)}{h}g(s)ds\\ &\qquad + \frac{1}{h}\int_t^{t+h}\mathcal{P}_\alpha(t+h-s)g(s)ds\end{aligned}$$

From (3.4) and (H), the integral $\int_0^t \mathcal{P}'_\alpha(t-s)(g(s)-g(t))\,ds$ is absolutely convergent. By applying the Dominated Convergence Theorem (Lebesgue's Dominated Convergence Theorem), we obtain:

$$\begin{aligned}&\lim_{h\to0}\int_0^t \frac{\mathcal{P}_\alpha(t+h-s)-\mathcal{P}_\alpha(t-s)}{h}(g(s)-g(t))ds\\ &\qquad = \int_0^t \mathcal{P}'_\alpha(t-s)(g(s)-g(t))ds\end{aligned}$$

From Lemma 2.2(i), we have:

$$\begin{aligned}&\int_0^t \frac{\mathcal{P}_\alpha(t+h-s)-\mathcal{P}_\alpha(t-s)}{h}g(t)ds\\ &= \frac{1}{h}\int_h^{t+h}\mathcal{P}_\alpha(s)g(t)ds - \frac{1}{h}\int_0^t \mathcal{P}_\alpha(s)g(t)ds\\ &= \frac{1}{h}\int_t^{t+h}\mathcal{P}_\alpha(s)g(t)ds - \frac{1}{h}\int_0^h \mathcal{P}_\alpha(s)g(t)ds\\ &\to (\mathcal{P}_\alpha(t)-I)\,g(t) \text{ khi } h\to0\end{aligned}$$

By combining these estimates and using the identity $\int_0^t \mathcal{P}'_\alpha(t-s)g(s)\,ds = (\mathcal{P}_\alpha(t)-I)\,g(t)$, we obtain:

$$\begin{aligned}&\lim_{h\to0}\int_0^t \frac{\mathcal{P}_\alpha(t+h-s)-\mathcal{P}_\alpha(t-s)}{h}g(s)ds\\ &\qquad = \int_0^t \mathcal{P}'_\alpha(t-s)g(s)ds\end{aligned}$$

Moreover

$$\begin{aligned}&\frac{1}{h}\int_0^h \mathcal{P}_\alpha(s)g(t+h-s)ds =\\ &\qquad \frac{1}{h}\int_0^h \mathcal{P}_\alpha(s)(g(t+h-s)-g(t))ds\\ &\qquad + \frac{1}{h}\int_0^h \mathcal{P}_\alpha(s)g(t)ds\end{aligned}$$

Thanks to the assumption on the function g, we have:

$$\left\|\frac{1}{h}\int_0^h \mathcal{P}_\alpha(s)(g(t+h-s)-g(t))ds\right\| \le CL_1h^\theta \to 0,$$

as $h \to 0$. From Lemma 2.2(i), we deduce that $\lim_{h\to 0} \frac{1}{h}\int_0^h \mathcal{P}_\alpha(s)g(t)\,ds = g(t)$. Therefore

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \mathcal{P}_\alpha(t+h-s)g(s)ds = g(t).$$

This means that $\Phi(t)$ is continuously differentiable at t and its derivative, denoted $\Phi'_+(t)$, satisfies:

$$\Phi'_+(t) = \int_0^t \mathcal{P}'_\alpha(t-s)g(s)ds + g(t).$$

By reasoning as above, assuming that $0 < h < t$, we can easily deduce that $\Phi(t)$ is differentiable at t_- and $\Phi'_+(t) = \Phi'_-(t)$. From Lemma 3.2 and (H), we obtain:

$$\begin{aligned}\Phi'(t) &= \int_0^t \mathcal{P}'_\alpha(t-s)g(s)ds + g(t)\\ &= \int_0^t \mathcal{P}'_\alpha(t-s)(g(s)-g(t))ds\\ &\quad + \int_0^t \mathcal{P}'_\alpha(t-s)g(t)ds + g(t)\\ &= w(t) + \mathcal{P}_\alpha(t)g(t) \in C\left((0,T];L^2(\Omega)\right).\end{aligned}$$

Now we show that $\Phi(t) \in D(A)$ for all $0 < t \le T$. To achieve this, we rewrite:

$$\begin{aligned}A\Phi(t) &= A\Phi_1(t) + A\Phi_2(t)\\ &= \int_0^t A\mathcal{P}_\alpha(t-s)(g(s)-g(t))ds\\ &\quad + \int_0^t A\mathcal{P}_\alpha(t-s)g(t)ds\end{aligned}$$

For a fixed $t \in (0,T]$, from Lemma 2.4 and (H), these two integrals converge absolutely. Therefore, $\Phi(t) \in D(A)$. By repeating the reasoning used in the proof of Lemma 3.2, we have $A\Phi_1 \in C\left((0,T];L^2(\Omega)\right)$. To prove a similar conclusion for $A\Phi_2(t)$, choose h such that $\epsilon \le t < t+h \le T$, and note that

$$\begin{aligned}&A\Phi_2(t+h) - A\Phi_2(t)\\ &= \int_0^h A\mathcal{P}_\alpha(t+h-s)g(t+h)ds\\ &\quad + \int_h^{t+h} A\mathcal{P}_\alpha(t+h-s)g(t+h)ds\\ &\quad - \int_0^t A\mathcal{P}_\alpha(t-s)g(t)ds\\ &= \int_0^h A\mathcal{P}_\alpha(t+h-s)g(t+h)ds\\ &\quad + \int_0^t A\mathcal{P}_\alpha(t-s)(g(t+h)-g(t))ds\end{aligned}$$

From Lemma 2.4 and (H), it is easy to deduce that

$$\begin{aligned}&\|A\Phi_2(t+h) - A\Phi_2(t)\|\\ &\le C\|g(t+h)\|\int_0^h (t+h-s)^{\alpha-1}ds\\ &\quad + CL_1h^\theta \int_0^t (t-s)^{\alpha-1}ds\\ &\le C\sup_{s\in[\epsilon,T]}\|g(s)\|\epsilon^{\alpha-1}h + CL_1h^\theta\frac{T^\alpha}{\alpha}. \qquad (3.5)\end{aligned}$$

This means that $A\Phi_2 \in C\left([\epsilon,T];L^2(\Omega)\right)$. Therefore, $A\Phi_2 \in C\left((0,T];L^2(\Omega)\right)$ because ϵ is arbitrary. Next, we prove that $\partial_t^\alpha A\Phi \in C\left((0,T];L^2(\Omega)\right)$. From Lemma 2.2(ii), we have:

$$\begin{aligned}-\left(1+k\partial_t^\alpha\right)A\Phi(t) &= -\frac{d}{dt}(\eta * A\Phi(t))\\ &= \frac{d}{dt}\left((\mathcal{P}_\alpha(t)-I)*g\right)\\ &= \Phi'(t) - g(t).\end{aligned}$$

Here, * denotes the convolution operator. Since $A\Phi = A\Phi_1 + A\Phi_2 \in C\left((0,T];L^2(\Omega)\right)$ has been proven, we conclude that $\partial_t^\alpha A\Phi \in C\left((0,T];L^2(\Omega)\right)$. Moreover, $u(t) = \mathcal{P}_\alpha(t)u_0 + \Phi(t)$ is a classical solution of (3.1).

According to Lemma 2.3, we obtain that $\mathcal{P}'_\alpha(t)u_0$ and $A\mathcal{P}_\alpha(t)u_0$ are Lipschitz continuous on $(0,T]$. From Lemma 3.2 and (H), we have:

$$\begin{aligned}\Phi'(t) &= \int_0^t \mathcal{P}'_\alpha(t-s)(g(s)-g(t))ds\\ &\quad + \mathcal{P}_\alpha(t)g(t) \in C^\theta\left((0,T];L^2(\Omega)\right)\end{aligned}$$

This follows from similar reasoning used in the proof of Lemma 3.2 and equation (3.5), which shows that $A\Phi = A\Phi_1 + A\Phi_2 \in C^\theta\left((0,T];L^2(\Omega)\right)$. Therefore, $\partial_t u, Au \in C^\theta\left((0,T];L^2(\Omega)\right)$, and $k\partial_t^\alpha Au = g(t) - \partial_t u - Au \in C^\theta\left((0,T];L^2(\Omega)\right)$. This completes the proof. □

4 CONCLUSION

The Rayleigh-Stokes problem for certain non-Newtonian fluids has received considerable attention due to its practical importance. This unsteady flow problem examines the motion of a fluid flow caused by a sudden displacement of the boundary. The mathematical models are derived by considering the constitutive relationship of viscoelastic fluids with fractional derivatives.

In this paper, we have studied the Rayleigh-Stokes problem for generalized second-order fluids. Our goal is to establish the fundamental theory of solutions to this equation. In particular, the existence and regularity of classical solutions are investigated. The proof of the main results relies on the fixed-point technique and estimates for the resolvent operator.

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