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A RESULT ON THE FINITENESS OF COASSOCIATED PRIMES OF LOCAL HOMOLOGY WITH RESPECT TO A PAIR OF IDEALS

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Abstract:

We introduce the concept local homology with respect to a pair of ideals, which is dual to the generalized local cohomology in (Takahashi, Yoshino, & Yoshizawa, 2009) and extension from the local homology module in (Cuong & Nam, 2001). We also study about some properties of these modules and give a result on the finiteness of coassociated primes of these modules.

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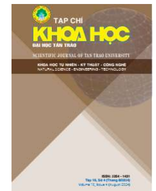

MỘT KẾT QUẢ VỀ TÍNH HỮU HẠN CỦA TẬP IĐÊAN NGUYÊN TỐ ĐỐI LIÊN KẾT CỦA MÔĐUN ĐỒNG ĐIỀU ĐỊA PHƯƠNG THEO MỘT CẶP IĐÊAN

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Từ khóa:
môđun compact tuyến tính, môđun đồng điều địa phương, môđun đối đồng điều địa phương, môđun CFA.

Tóm tắt:

Trong bài báo này, chúng tôi sẽ giới thiệu về môđun đồng điều địa phương theo một cặp iđêan, đây là một khái niệm đối ngẫu với khái niệm về môđun đối đồng điều địa phương suy rộng trong (Takahashi, Yoshino, & Yoshizawa, 2009) và mở rộng từ khái niệm về môđun đồng điều địa phương trong (Cuong & Nam, 2001). Chúng tôi cũng nghiên cứu về một số tính chất cơ bản và đưa ra một kết quả về tính hữu hạn của tập iđêan nguyên tối đối liên kết của lớp môđun này.

1 INTRODUCTION

The theory of local cohomology has existed and developed strongly for more than 50 years and proved to be a very important tool in commutative algebra. There have been many extensions from this theory. Takahashi, Yoshino and Yoshizawa (2009) introduced the definition of local cohomology with respect to a pair of ideals, which is a generalization of local cohomology modules of Grothendieck. Let R is a notherian communicative ring, I, J are ideals of R and M is an $R-$module. The $i-$th local cohomology module with respect to a pair of ideals (I, J), $H^i_{I,J}(M)$, defined by

$$H^i_{I,J}(M) = R^i\Gamma_{I,J}(M)$$

where $\Gamma_{I,J}(M)$ is the set of elements x of M such that $I^n x \subseteq Jx$ for some integer n. Moreover, they found the relation between $H^i_{I,J}(M)$ and $H^i_I(M)-$local cohomology for a ideal through the isomorphism

$$H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M)$$

in which $\widetilde{W}(I, J)$ the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a}+J$ for some integer n and partial order on $\widetilde{W}(I, J)$ by letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{b} \subseteq \mathfrak{a}$ for $\mathfrak{a}, \mathfrak{b} \in \widetilde{W}(I, J)$.

By duality, we introduce the definition of local homology for a pair of ideals (I, J), denote by $H^{I,J}_i(M)$, as follow

$$H^{I,J}_i(M) = \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Tor}^R_i(R/\mathfrak{a}, M).$$

Besides, we show some basic properties of these modules of linearly compact modules. An interesting problem in commutative algebra is the finiteness of coassociated primes of local homology. Yassemi (Yassemi, 1995) defined the set of *coassociated prime ideals* of an R-module M, and denoted by $\operatorname{Coass}_R(M)$, to be the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}_R L = \mathfrak{p}$. N. M. Tri (Tri, 2021) gave the concept of CFA modules and used it as a tool to study about the fininess of coassociated primes of local homology modules. An R-module M is called CFA if there is a submodule N such that $\operatorname{Cosupp}_R N$ is a finite set and M/N is an artinian $R-$module. And the author also proved that "If M is a CFA $R-$module, then $\operatorname{Coass}_R M$ is a finite set". The last theorem of this paper give a similar result for the local homology with respect to a pair of ideals.

2 MAIN RESULTS

Throughout this paper, R will always be a commutative noetherian ring with non-zero identity. Firstly, we recall the concept of linearly compact modules by using the terminology of I. G. Macdonald (Macdonald, 1962). Let M be a topological R-module. A *nucleus* of M is a neighborhood of the zero element of M, and a *nuclear base* of M is a base for the nuclei of M. If N is a submodule of M which contains a nucleus then N is open in M and M/N is discrete. M is Hausdorff if and only if the intersection of all the nuclei of M is 0. M is said to be *linearly topologized* if M has a nuclear base consisting of submodules. A Hausdorff linearly topologized R-module M is said to be *linearly compact* if M has the following property: if $\mathcal{F}$ is a family of closed cosets (i.e. cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

Next, let I, J be two ideals of the noetherian ring R and M an $R-$module. Denote by $\widetilde{W}(I, J)$ the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a}+J$ for some integer n (Takahashi, Yoshino, Yoshizawa, 2009). We define a partial order on $\widetilde{W}(I, J)$ by letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{b} \subseteq \mathfrak{a}$ for $\mathfrak{a}, \mathfrak{b} \in \widetilde{W}(I, J)$. If $\mathfrak{a} \leq \mathfrak{b}$ we have the homomorphisms $\operatorname{Tor}^R_i(R/\mathfrak{b}^t, M) \longrightarrow \operatorname{Tor}^R_i(R/\mathfrak{a}^t, M)$ for all $t > 0$ and $i \geq 0$. It induces a homomorphism of local homology modules $H^{\mathfrak{b}}_i(M) \longrightarrow H^{\mathfrak{a}}_i(M)$. Hence we have an inverse system of modules $\{H^{\mathfrak{a}}_i(M)\}_{\mathfrak{a}\in\widetilde{W}(I,J)}$. We suggest the following definition.

Definition 2.1. Let I, J be two ideals of the ring R and M an $R-$module. The $i-$th local homology module $H^{I,J}_i(M)$ of M with respect to a pair of ideals (I, J) is defined by

$$H^{I,J}_i(M) = \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^{\mathfrak{a}}_i(M).$$

Proposition 2.2. Let M be a linearly compact $R-$module. Then for all $i \geqslant 0$, $H^{I,J}_i(M)$ is a linearly compact $R-$module.

Proof. It follows from (Cuong & Nam, 2008) that $\{H^{\mathfrak{a}}_i(M)\}_{a\in\widetilde{W}(I,J)}$ forms an inverse system of linearly compact modules with continuos homomorphisms. Hence $H^{I,J}_i(M)$ is also a linearly compact $R-$module by (Macdonald, 1973).

The following proposition shows that local homology with respect to a pair of ideals modules can be commuted with inverse limits of inverse systems of linearly compact $R-$modules.

Proposition 2.3. Let $\{M_t\}$ be an inverse system of linearly compact $R-$modules with the continuous homomorphisms. Then

$$H_i^{I,J}(\varprojlim_t M_t) \cong \varprojlim_t H_i^{I,J}(M_t).$$

Proof. Note that inverse limits are commuted and by (Cuong & Nam, 2008) we have

$$\begin{aligned} H_i^{I,J}(\varprojlim_t M_t) &= \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H_i^{\mathfrak{a}}(\varprojlim_t M_t) \\ &\cong \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \varprojlim_t (H_i^{\mathfrak{a}}(M_t)) \\ &\cong \varprojlim_t \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H_i^{\mathfrak{a}}(M_t) \\ &= \varprojlim_t H_i^{I,J}(M_t). \end{aligned}$$

Lemma 2.4. Let M be an linearly compact $R-$module. Then for all $j \geqslant 0$,

$$H_i^{I,J}(H_j^{I,J}(M)) \cong \begin{cases} H_j^{I,J}(M), & i=0 \\ 0, & i>0 \end{cases}$$

Proof. By the (Cuong & Nam, 2008), $H_j^{\mathfrak{a}}(M)$ is linearly compact for all $\mathfrak{a}\in\widetilde{W}(I,J)$. Then we have by (Cuong & Nam, 2008)

$$\begin{aligned} H_i^{I,J}(H_j^{I,J}(M)) &= \varprojlim_{\mathfrak{b}\in\widetilde{W}(I,J)} H_i^{\mathfrak{b}}(\varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H_j^{\mathfrak{a}}(M)) \\ &\cong \varprojlim_{\mathfrak{b}\in\widetilde{W}(I,J)} \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H_i^{\mathfrak{b}}(H_j^{\mathfrak{a}}(M)) \\ &\cong \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \varprojlim_{\mathfrak{b}\in\widetilde{W}(I,J)} H_i^{\mathfrak{b}}(H_j^{\mathfrak{a}}(M)) \end{aligned}$$

From 2.7 (Cuong & Nam, 2008), we have

$$\begin{aligned} H_i^{\mathfrak{b}}(H_j^{\mathfrak{a}}(M)) &= \varprojlim_t \operatorname{Tor}_i^R(R/\mathfrak{b}^t, \varprojlim_s(\operatorname{Tor}_j^R(R/\mathfrak{a}^s, M))) \\ &\cong \varprojlim_t \varprojlim_s \operatorname{Tor}_i^R(R/\mathfrak{b}^t, \operatorname{Tor}_j^R(R/\mathfrak{a}^s, M)) \\ &\cong \varprojlim_s \varprojlim_t \operatorname{Tor}_i^R(R/\mathfrak{b}^t, \operatorname{Tor}_j^R(R/\mathfrak{a}^s, M)) \end{aligned}$$

Let $\underline{x} = (x_1, \ldots, x_r)$ is a system of generator of $\mathfrak{b}$ and $\underline{x}(t) = (x_1^t, \ldots, x_r^t)$. It follows from (Cuong & Nam, 2008) that

$$H_i^{\mathfrak{b}}(H_j^{\mathfrak{a}}(M)) \cong \varprojlim_s \varprojlim_t H_i(\underline{x}(t), \operatorname{Tor}_j^R(R/\mathfrak{a}^s, M))$$

Then for all $\mathfrak{b} \geqslant \mathfrak{a}$, we have $\underline{x}(t) \subseteq \mathfrak{b}^t \subseteq \mathfrak{a}^t \subseteq \mathfrak{a}^s$, for all $t \geqslant s$. Hence $\underline{x}(t)\operatorname{Tor}_j^R(R/\mathfrak{a}^s, M) = 0, \forall t \geqslant s$. Then we have

$$\varprojlim_t H_0(\underline{x}(t), \operatorname{Tor}_j^R(R/\mathfrak{a}^s, M)) \cong \operatorname{Tor}_j^R(R/\mathfrak{a}^s, M)$$

and

$$\varprojlim_t H_i(\underline{x}(t), \operatorname{Tor}_j^R(R/\mathfrak{a}^s, M)) = 0, \forall i > 0.$$

So we have $H_i^{I,J}(H_j^{I,J}(M)) = 0$ for all $i > 0$. When $i = 0$, we proved $H_0^{\mathfrak{b}}(H_j^{\mathfrak{a}}(M)) \cong H_j^{\mathfrak{a}}(M)$ for all $\mathfrak{b} \geqslant \mathfrak{a}$. Therefore, $H_0^{I,J}(H_j^{I,J}(M)) \cong \varprojlim_{\mathfrak{a}\in\widetilde{W}(I,J)} \varprojlim_{\mathfrak{b}\in\widetilde{W}(I,J)} H_j^{\mathfrak{a}}(M) = H_j^{I,J}(M)$.

Lemma 2.5. Let M be an linearly compact $R-$module. Then

$$H_i^{I,J}(\bigcap_{\mathfrak{a}\in\widetilde{W}(I,J)} \mathfrak{a}\, M) \cong \begin{cases} 0, & i=0 \\ H_i^{I,J}(M), & i>0 \end{cases}.$$

Proof. For each $\mathfrak{a}\in\widetilde{W}(I,J)$, we have short exact sequence of linearly compact $R-$module

$$0 \to \mathfrak{a}^t M \to M \to M/\mathfrak{a}^t M \to 0,$$

for all $t > 0$, because $\{\mathfrak{a}^t M\}$ is inverse system of linearly compact modules, by (Cuong & Nam, 2008) we have a short exact sequence of linearly compact modules

$$0 \to \bigcap_{t>0} \mathfrak{a}^t M \to M \to \Lambda_{\mathfrak{a}}(M) \to 0.$$

From that, we have sequence

$$0 \to \bigcap_{\mathfrak{b}\in\widetilde{W}(I,J)} \mathfrak{b}\, M \to M \to \Lambda_{I,J}(M) \to 0.$$

Hence we get a long exact sequence of local homology modules

$$\begin{aligned} \cdots \to H_{i+1}^{I,J}(\Lambda_{I,J}(M)) \to H_i^{I,J}(\bigcap_{\mathfrak{b}\in\widetilde{W}(I,J)} \mathfrak{b}\, M) \to \\ \to H_i^{I,J}(M) \to H_i^{I,J}(\Lambda_{I,J}(M)) \to \cdots \\ \to H_1(\Lambda_{I,J}(M)) \to H_0^{I,J}(\bigcap_{\mathfrak{b}\in\widetilde{W}(I,J)} \mathfrak{b}\, M) \to \\ \to H_0^{I,J}(M) \to H_0^{I,J}(\Lambda_{I,J}(M)) \to 0. \end{aligned}$$

The lemma now follows from Lemma 2.4.

The following result gives us a condition for the finiteness of coassociated primes of local homology with respect to a pair of ideals.

Theorem 2.6. Let M be a linearly compact $R-$module and t a non-negative integer. If M and $H_i^{I,J}(M)$ are CFA for all $i < t$, then there is an ideal $\mathfrak{a}\in\widetilde{W}(I,J)$ such that

$$R/\,\mathfrak{a} \otimes_R H_t^{I,J}(M)$$

is CFA. In particular, $\operatorname{Coass}_R(H_t^{I,J}(M))$ is finite.

Proof. The proof is by induction on t. Let $t = 0$. There is an ideal $\mathfrak{a} \in \tilde{W}(I, J)$ such that $\cap \mathfrak{a} M = \mathfrak{a} M$. The short exact sequence

$$0 \to \mathfrak{a} M \to M \to H_0^{I,J}(M) \to 0$$

induces the following exact sequence

$$R/\mathfrak{a} \otimes_R \mathfrak{a} M \to R/\mathfrak{a} \otimes_R M \to R/\mathfrak{a} \otimes_R H_0^{I,J}(M) \to 0.$$

By [11], $R/\mathfrak{a} \otimes_R H_0^{I,J}(M)$ is CFA.

Let $t > 0$. Since $H_t^{I,J}(M) \cong H_t^{I,J}(\cap \mathfrak{a} M) = H_t^{I,J}(\mathfrak{a} M)$, we can replace M by $N = \mathfrak{a} M$. It implies that $\mathfrak{a} N = N$. Hence, there is an element $x \in \mathfrak{a}$ such that $xN = N$. The short exact sequence

$$0 \to 0 :_N x \to N \xrightarrow{x} N \to 0$$

induces a long exact sequence

$$\cdots \to H_t^{I,J}(N) \xrightarrow{x} H_t^{I,J}(N) \xrightarrow{\alpha} H_{t-1}^{I,J}(0 :_N x) \to$$
$$\xrightarrow{\beta} H_{t-1}^{I,J}(N) \to \cdots \to H_0^{I,J}(N) \to 0.$$

By the assumption, $H_i^{I,J}(0 :_N x)$ is CFA for all $i < t-1$. The exact sequences

$$H_t^{I,J}(N) \xrightarrow{x} H_t^{I,J}(N) \to \operatorname{Im} \alpha \to 0$$

and

$$0 \to \operatorname{Im} \alpha \to H_{t-1}^{I,J}(0 :_N x) \to \operatorname{Im} \beta \to 0$$

lead the isomorphism

$$R/\mathfrak{a} \otimes H_t^{I,J}(N) \cong R/\mathfrak{a} \otimes \operatorname{Im} \alpha$$

and the long exact sequence

$$\operatorname{Tor}_1^R(R/\mathfrak{a}, \operatorname{Im} \beta) \to R/\mathfrak{a} \otimes \operatorname{Im} \alpha \to$$
$$\to R/\mathfrak{a} \otimes H_{t-1}^{I,J}(0 :_N x).$$

It follows from (Tri, 2021) that $\operatorname{Im} \beta$ is CFA. Therefore, so is $\operatorname{Tor}_1^R(R/\mathfrak{a}, \operatorname{Im} \beta)$. The inductive hypothesis shows that $R/\mathfrak{a} \otimes H_{t-1}^{I,J}(0 :_N x)$ is a CFA module. Therefore, we have $R/\mathfrak{a} \otimes \operatorname{Im} \alpha$ is CFA and this completes the proof.

3 CONCLUSION

In this paper, we gave the concept of local homology with respect to a pair of ideals. This is a fairly new class of modules and there are still many aspects to research. We also showed some properties of these modules and gave a result on the finitness of coassociated primes of these modules.

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