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THE MEAN VALUE THEOREM FOR HOLOMORPHIC FUNCTIONS OF A COMPLEX VARIABLE

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Abstract

The mean value theorem is one of the most fundamental results in real analysis. However, it fails for holomorphic function of a complex variable even if the function is differentiable on whole complex plane. In this paper, we clarify some throughout examples, and we also show the equivalence between the complex version of Rolle's theorem and Lagrangle's mean value theorem.

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ĐỊNH LÝ GIÁ TRỊ TRUNG BÌNH ĐỐI VỚI HÀM GIẢI TÍCH MỘT BIẾN PHỨC

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Thông tin bài viết
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Từ khoá:
Định lý giá trị trung bình, Định lý giá trị trung bình trong ℂ, Định lý Rolle trong ℂ, Định lý Fellt, Định lý Myers.

Tóm tắt
Định lý giá trị trung bình là một kết quả cơ bản của giải tích thực. Tuy nhiên nó không còn đúng cho hàm chỉnh hình một biến phức, kể cả trong trường hợp hàm là khả vi trong toàn mặt phẳng phức, bài báo làm rõ điều này bằng một số ví dụ cụ thể, đồng thời chỉ ra sự tương đương giữa Định lý Rolle và Định lý giá trị trung bình trong mặt phẳng phức.

1. Introduction

It is known that the mean value theorem is one of the most fundamental results in analysis. Some deformations of this theorem were shown, such as Flett's theorem [1858], Myers's theorem [1977], Sahoo và Riedel's theorem [1998]. These theorems have many applications in analysis, in solving equations, systems of equations, finding solutions or breakpoints of polynomials, used to solve many optimization problems, economy,... [3].

Let f be a continuous function on a closed interval $[a,b]$. The difference between the values of f at the endpoints of $[a,b]$, if the derivative $f'(a)$ exists, and one gets:

$$f(b)-f(a)\approx f'(a).(b-a) \qquad (1)$$

Set $\Delta x=b-a$, then (1) can be written as:

$$f(a+x)\approx f(a)+f'(a).\Delta x \qquad (2)$$

Formula (1) can be replaced by:

$$f(b)-f(a)=f'(c).(b-a) \qquad (3)$$

where $c \in (a,b)$, the function f is differentiable at every point of (a,b). This is the result which known as the Mean value theorem.

In the complex analysis, a natural question arises: Does the mean value theorem still hold in the field of complex numbers?

The results of the paper is developped based on the two papers [1], [5] and [7]. Its content analyzes carefully some examples showing that, mean value theorem and the some theorems were developed from it fails for complex valued functions, even if the function is differentiable on whole the complex plane. At the same time, the equivalence of Rolle's and Mean value theorems in the complex plane are proved in detail, many calculations are more detailed than the original document.

2. Content

a) Mean value theorem and some its deformations

Theorem 1. (Largrange's mean value theorem)

Let f be real continuous function on $[a,b]$ and differentiable in (a,b). There exists a point $c \in (a,b)$ such that

$$f(b)-f(a)=f'(\mathrm{c}).(b-a). \qquad (4)$$

If $f(a)=f(b)$, then the mean value theorem reduces to Rolle's theorem which is also the another most fundamental result in real analysis.

Theorem 2. (Rolle's theorem)

Let f be a real continuous function on $[a,b]$ and differentiable in (a,b). Furthermore, assume $f(a)=f(b)$. Then there is a point $c \in (a,b)$ such that

$$f'(c)=0. \qquad (5)$$

Rolle's theorem and Largrange's mean value theorem in real-valued functions are equivalent [5].

Proof of the equivalence.

i) It is easy, Theorem 1 deduce Theorem 2.

Suppose f satisfy the conditions of Theorem 1. Then $f(b)-f(a)=f'(\mathrm{c})(b-a)$. By $f(a)=f(b)$ so $f'(\mathrm{c})(b-a)=0$. It deduce $f'(c)=0$.

ii) We will prove Theorem 2 deduce Theorem 1. Indeed:

Suppose f satisfy the conditions of Theorem 2.

Consider the auxiliary function:

$$g(x)=\begin{vmatrix} f(x) & f(a) & f(b) \\ x & a & b \\ 1 & 1 & 1 \end{vmatrix}$$

$$=(a-b)f(x)-(x-b)f(a)+(x-a)f(b).$$

We see that $g(x)$ is a real function: continuous on the closed interval $[a,b]$, differentiable in the open interval (a,b) and $g(a)=g(b)$. Then, by Theorem 2: there exists a point $c \in (a,b)$ such that $g'(c)=0$. More $g'(x)=(a-b)f'(x)-f(a)+f(b)$.

Hence,

$$g'(c)=(a-b)f'(c)-f(a)+f(b)=0$$

$$\Leftrightarrow f(b)-f(a)=f'(c)(b-a).$$

Which proves that Theorem 2 deduce Theorem 1. So, two theorems are equivalent.

Next, we consider some theorems developed from the Mean value theorem.

Theorem 3. (Flett's Mean Value Theorem [2]).

Let $f:[a,b]\to\mathbb{R}$ *be differentiable on* $[a,b]$ *and* $f'(a)=f'(b)$. *Then there exists a point* $c\in(a,b)$ *such that*

$$f(c)-f(a)=f'(c).(c-a). \qquad (6)$$

Theorem 4. (Myers's Mean Value Theorem [4]).

Let $f:[a,b]\to\mathbb{R}$ *be differentiable on* $[a,b]$ *and* $f'(a)=f'(b)$. *Then there exists a point* $c\in(a,b)$ *such that*

$$f(b)-f(c)=f'(c).(b-c). \qquad (7)$$

Theorem 5. (Sahoo và Riedel's Theorem [6])

Let $f:[a,b]\to\mathbb{R}$ *be differentiable on* $[a,b]$. *Then there exists a point* $c\in(a,b)$ *such that*

$$f(c)-f(a)$$
$$=f'(c)(c-a)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2. \quad (8)$$

Remark:

Myers's theorem is a complete complement to Flett's theorem.

When $f'(a)=f'(b)$ then Sahoo and Riedel's theorem become Flett's theorem.

By analysing some examples in both real and complex analysis, we show that the Mean value theorem and some its deformations do not hold for holomorphic functions of one complex variable. Such as:

For the case of Flett's theorem, consider the function $f(z)=e^z-z,\ z\in\mathbb{C}$. We see that f is holomorphic and $f'(z)=e^z-1$, therefore $f'(2k\pi i)=e^{2k\pi i}-1=0\ \ \forall k\in\mathbb{Z}$. Consider the closed interval $[0,2\pi i]$, we have $f'(0)=f'(2\pi i)$.

Nevertheless, $f(z)-f(0)=f'(z)(z-0)=f'(z)z$ has no solution in the interval $(0,2\pi i)$. Indeed, the equation

$$f(z)-f(0)=f'(z)(z-0)\Leftrightarrow 1-z=e^{-z}.$$

When $z=iy$, we have:

$1-iy=e^{-iy}=\cos y-i\sin y$, the comparison of the real and imaginary parts gives:

$$\begin{cases}\cos y=1\\ \sin y=y\end{cases}$$

Evidently, this system of equations has no solution in the interval $(0,2\pi)$. This means $f(z)-f(0)=f'(z)(z-0)$ has no solution in the interval $(0,2\pi i)$.

Thus Flett's theorem fails in the complex domain.

For the case of Largrange's Mean Value Theorem, consider the function $f(z)=\exp\left\{i\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right\}$, z_1,z_2 are two distinct points in the complex plane. We can calculate:

$$f(z_1)=\exp\left\{i\frac{2z_1-(z_1+z_2)}{z_2-z_1}\pi\right\}=e^{-i\pi}=-1;$$

$$f(z_2)=\exp\left\{i\frac{2z_2-(z_1+z_2)}{z_2-z_1}\pi\right\}=e^{i\pi}=-1;$$

$$f'(z)=\exp\left\{i\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right\}.\frac{2i\pi}{z_2-z_1}\neq 0,$$

$\forall z\in\mathbb{C}.$

Therefore

$$\begin{aligned}&f(z_2)-f(z_1)-f'(z)(z_2-z_1)\\&=-1+1-f'(z)(z_2-z_1)\\&=-f'(z)(z_2-z_1)\neq 0,\forall z\in\mathbb{C}.\end{aligned}$$

So $f(z_2)-f(z_1)=f'(z)(z_2-z_1)$ has no solution. Thus Largrange's mean value theorem fails in the complex domain.

For the case of Rolle's theorem, consider the function $f(z)=e^z-1,\ z\in\mathbb{C}$. We see that $f(2k\pi i)=0,\ \forall k\in\mathbb{Z}$ but $f'(z)=e^z=0$ has no solution in the complex plane. Thus Rolle's theorem fails in the complex domain.

The previous examples show that the Mean value theorem does not hold in the complex plane if we fix the hypothesis as in the case of real analysis. Next, we present the similar theorems which hold in complex analysis.

b) Rolle's and Mean value theorem in complex plane [1],[7]

Let a and b be distinct points in $\mathbb{C}$. Denote $[a,b]$ the set $\{a+t(b-a):t\in[0,1]\}$ and we will refer to $[a,b]$ as a line segment or a closed interval in $\mathbb{C}$. Similarly, (a,b) denotes the set $\{a+t(b-a):t\in(0,1)\}$.

Theorem 6. (Complex Rolle's theorem)

Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f satisfy $f(a)=f(b)=0$. Then there exist $z_1,z_2\in(a,b)$ such that $\operatorname{Re}(f'(z_1))=0$ and $\operatorname{Im}(f'(z_2))=0$.

<u>*Proof*</u>.

Let

$$a=a_1+ia_2,\ b=b_1+ib_2,$$
$$f(z)=u(z)+iv(z)\ \ \forall z\in D_f$$

and

$$\phi(t)=(b_1-a_1)u(a+t(b-a))+(b_2-a_2)v(a+t(b-a)),t\in[0,1].$$

By hypothesis $f(a)=f(b)=0$ so

$$u(a)=v(a)=u(b)=v(b)=0.$$

We can calculate:

$$\phi(0)=(b_1-a_1)u(a)+(b_2-a_2)v(a)=0;$$
$$\phi(1)=(b_1-a_1)u(b)+(b_2-a_2)v(b)=0.$$

According to Rolle's theorem, it exists $t_1\in(0,1)$ such that $\phi'(t_1)=0$.

Let $z_1=a+t_1(b-a)$. We have

$$\begin{aligned}0=\phi'(t_1)&=(b_1-a_1)\left[\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}\right]\\&\quad+(b_2-a_2)\left[\frac{\partial v}{\partial x}.\frac{dx}{dt}+\frac{\partial v}{\partial y}.\frac{dy}{dt}\right]\\&=(b_1-a_1)\left[\frac{\partial u}{\partial x}(z_1).(b_1-a_1)+\frac{\partial u}{\partial y}(z_1).(b_2-a_2)\right]\\&\quad+(b_2-a_2)\left[\frac{\partial v}{\partial x}(z_1).(b_1-a_1)+\frac{\partial v}{\partial y}(z_1).(b_2-a_2)\right]\\&=(b_1-a_1)^2.\frac{\partial u}{\partial x}(z_1)+(b_1-a_1).(b_2-a_2).\frac{\partial u}{\partial y}(z_1)\\&\quad+(b_2-a_2).(b_1-a_1).\frac{\partial v}{\partial x}(z_1)+(b_2-a_2)^2.\frac{\partial v}{\partial y}(z_1)\\&=\left[(b_1-a_1)^2+(b_2-a_2)^2\right]\frac{\partial u}{\partial x}(z_1).\end{aligned}$$

So $\operatorname{Re}(f'(z_1))=\dfrac{\partial u}{\partial x}(z_1)=0.$

Apply to the function $g=-if$, it exists $z_2\in(a,b)$ such that:

$$0=\operatorname{Re}(g'(z_2))=\frac{\partial v}{\partial x}(z_2)=-\frac{\partial u}{\partial y}(z_2)=\operatorname{Im}(f'(z_2)).$$

Theorem 7. (Complex Mean value theorem)

Let f be a holomerphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f. Then there exist $z_1,z_2\in(a,b)$ such that

$$\operatorname{Re}(f'(z_1))=\operatorname{Re}\left(\frac{f(b)-f(a)}{b-a}\right)$$

and

$$\operatorname{Im}(f'(z_2))=\operatorname{Im}\left(\frac{f(b)-f(a)}{b-a}\right).$$

Proof.

Let

$$g(z) = f(z) - f(a) - \frac{f(b)-f(a)}{b-a}(z-a), \quad \forall z \in D_f.$$

It is easy to see $g(a) = g(b)$.

By Theorem 6, there exists $z_1, z_2 \in (a,b)$ such that $\mathrm{Re}(g'(z_1)) = 0$ and $\mathrm{Im}(g'(z_2)) = 0$.

Furthermore

$g'(z) = f'(z) - \dfrac{f(b)-f(a)}{b-a}$, $\forall z \in D_f$. So

$$0 = \mathrm{Re}(g'(z_1)) = \mathrm{Re}(f'(z_1)) - \mathrm{Re}\left(\frac{f(b)-f(a)}{b-a}\right)$$

$$\Rightarrow \mathrm{Re}(f'(z_1)) = \mathrm{Re}\left(\frac{f(b)-f(a)}{b-a}\right);$$

$$0 = \mathrm{Im}(g'(z_2)) = \mathrm{Im}(f'(z_2)) - \mathrm{Im}\left(\frac{f(b)-f(a)}{b-a}\right)$$

$$\Rightarrow \mathrm{Im}(f'(z_2)) = \mathrm{Im}\left(\frac{f(b)-f(a)}{b-a}\right).$$

We see Theorem 6 and Theorem 7 are equivalent. Indeed:

i) Theorem 7 implies Theorem 6: Assume that f function satisfy the conditions of Theorem 7. Then, we have

$$\mathrm{Re}(f'(z_1)) = \mathrm{Re}\left(\frac{f(b)-f(a)}{b-a}\right)$$

and

$$\mathrm{Im}(f'(z_2)) = \mathrm{Im}\left(\frac{f(b)-f(a)}{b-a}\right),$$

$z_1, z_2 \in (a,b)$. Moreover $f(a) = f(b) = 0$ therefore $\mathrm{Re}\left(\dfrac{f(b)-f(a)}{b-a}\right) = 0$ and $\mathrm{Im}\left(\dfrac{f(b)-f(a)}{b-a}\right) = 0$, hence $\mathrm{Re}(f'(z_1)) = 0$ and $\mathrm{Im}(f'(z_2)) = 0$.

ii) Theorem 6 implies Theorem 7.

Assume that f satisfy the conditions of Theorem 6.

Consider the auxiliary function:

$$g(z) = \frac{1}{a-b}\begin{vmatrix} f(z) & f(a) & f(b) \\ z & a & b \\ 1 & 1 & 1 \end{vmatrix}$$

$$= f(z) - f(a)\frac{z-b}{a-b} + f(b)\frac{z-a}{a-b}. \qquad (9)$$

We see that $g(z)$ is a holomorphic function in D_f and satisfy the condition $g(a) = g(b) = 0$. Then, by Theorem 6, exists $z_1, z_2 \in (a,b)$ such that $\mathrm{Re}(f'(z_1)) = 0$ and $\mathrm{Im}(f'(z_2)) = 0$. From (9) we have:

$g'(z) = f'(z) - \dfrac{f(b)-f(a)}{b-a}$, $\forall z \in D_f$, so

$$0 = \mathrm{Re}(g'(z_1)) = \mathrm{Re}(f'(z_1)) - \mathrm{Re}\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \mathrm{Re}(f'(z_1)) = \mathrm{Re}\frac{f(b)-f(a)}{b-a};$$

$$0 = \mathrm{Im}(g'(z_2)) = \mathrm{Im}(f'(z_2)) - \mathrm{Im}\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \mathrm{Im}(f'(z_2)) = \mathrm{Im}\frac{f(b)-f(a)}{b-a}.$$

Then, Theorem 6 implies Theorem 7.

Therefore, Theorem 6 and Theorem 7 are equivalent.

3. Conclusion

By studying the results, the authors had to clarify the picture of *Mean Value Theorem in Complex plane.* By illustrating in details some examples, we showed that, Rolle's and Mean Value Theorem fails for holomorphic functions of a complex variable, even if the function is differentiable throughout the complex plane. The proof of the theorems were presented in details. In particular, the paper proved the equivalent between Rolle's theorem and Largrange's Mean value theorem, in which the calculation in details added.

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