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AN ITERATIVE METHOD FOR SOLVING A SPLIT VARIATIONAL INEQUALITY

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Abstract:

In this paper, we introduce two different interative methods for finding a solution of a split pseudomonotone variational inequality and a split feasibility problem in Hilbert spaces. The proposed algorithm is generated based on the subgradient extragradient method which requires only two projections at each iteration step and the second projection is conducted onto the half-space containing the constrained set. The strong convergence is proven with some mild conditions imposed on the operators as well as the parameters. A numerical result is provided at the end of the paper with the use of Python for the convergence illustration purpose of the studied method.

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MỘT PHƯƠNG PHÁP LẶP GIẢI BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN TÁCH

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Thông tin bài viết

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Từ khóa:
bài toán chấp nhận tách, bất đẳng thức biến phân tách, ánh xạ giả đơn điệu, phép chiếu metric, dưới đạo hàm tăng cường.

Tóm tắt:

Trong bài báo này, chúng tôi giới thiệu hai thuật toán lặp để tìm nghiệm của bài toán bất đẳng thức biến phân tách giả đơn điệu và bài toán chấp nhận tách trong không gian Hilbert. Phương pháp mà chúng tôi đề xuất được thiết lập dựa trên phương pháp dưới đạo hàm tăng cường trong đó người ta sử dụng hai phép chiếu cho mỗi bước lặp và ở phép chiếu thứ hai, sử dụng phép chiếu lên nửa không gian chứa miền ràng buộc. Sự hội tụ mạnh của thuật toán được chứng minh với một số giả thiết giảm nhẹ về tính đơnđiệu lên các toán tử cùng với một số điệu kiện đặt lên các dãy tham số. Một ví dụ số với kết quả thu được bằng ngôn ngữ lập trình Python nhằm minh họa cho sự hội tụ của thuật toán cũng được chúng tôi đưa ra ở cuối bài báo.

1 INTRODUCTION

Let C and Q be nonempty closed and convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $F_1 : H_1 \to H_1$ and $F_2 : H_2 \to H_2$ are mappings on H_1 and H_2, respectively. Let $B : H_1 \to H_2$ be a bounded linear operator and $B^* : H_2 \to H_1$ be the adjoint of B. The split variational inequality problem (SVIP) introduced firstly by Censor et al. [1] is formulated as

$$\begin{aligned} &\text{Find an element } x^* \in C : \\ &\langle F_1(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C, \end{aligned} \tag{1}$$

such that

$$y^* = Bx^* \in Q : \langle F_2(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Q. \tag{2}$$

Denote $\mathrm{Sol}(C, F_1)$ and $\mathrm{Sol}(Q, F_2)$ the solution sets of VIPs (1) and (2), respectively. Then the (SVIP) becomes a split feasibility problem (SFP) that is

$$\begin{aligned} &\text{Find } x^* \in \mathrm{Sol}(C, F_1) \text{ such that} \\ &y^* = Bx^* \in \mathrm{Sol}(Q, F_2). \end{aligned} \tag{3}$$

So (SFP) can be deduced from (SVIP) as a special case which has been investigated thoroughly to model the intensity-modulated radiation therapy (IMRT) and some other disciplines (see [2] - [4] for example).

To solve (SVIP) involving inverse strongly monotone mappings F_1, F_2, in [1], Censor and the coauthors used the projection method which was proven to be weakly convergent.

It is worth pointing that the projection method for monotone VIPs may fail to converge ([5], p. 1110). To deal with this obstacle, the extragradient method introduced by Korpelevich [6] for saddle problems can be applied to monotone VIPs to ensure the convergence. The extragradient method may be costly since finding the projection at each step of iteration is not easy. To overcome this difficulty Censor et al. [7] have modified the extragradient method for solving monotone VIPs by conducting the second projection onto a half-space containing the constrained set in stead of onto the last mentioned set. The method is called subgradident extragradient method. Censor and his colleagues also obtained the weak convergence of the modified method to the unique solution of the monotone VIP.

In 2017, Anh et al. [8] applied the subgradient extragradient method for solving a bilevel split pseudomonotone variational inequality problem (BSVIP) involving a strongly monotone mapping in the upper-level problem and pseudomonotone mappings in the lower-level one. The authors gained the strong convergence for the proposed schemes. Inspired by these results in [8], in this paper, we present the subgradient extragradient method for solving the split variational inequality problem (SVIP) involving pseudomonotone mappings and the split feasibility problem (SFP) as special cases of the methods proposed in [8].

The next parts of the paper is divided in what following sections. Section 2 is for the definitions, lemmas and other preliminaries that is needed in our convergence analysis. Section 3 is for the validity of the strong convergence of the proposed algorithms and the last section is devoted for a numerical example with the purpose of convergence illustration for the method considered in the previous section.

2 PRELIMINARIES

Let H be a real space with an inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a nonempty, closed and convex subset of H. Throughout this paper, with a sequence $\{x^k\}$ in H, we write $x^k \to x$ ($x^k \rightharpoonup x$) for the strong (weak) convergence of x^k to x. Recall that the metric projection from H onto C denoted by P_C is defined as

$$P_C(x) = \text{argmin}\{\|x - y\| : y \in C\}.$$

Lemma 2.1. *For given $x \in H$ and $y \in C$:*

(i) $y = P_C(x)$ if and only if

$$\langle x - y, z - y \rangle \leq 0, \forall z \in C.$$

(ii) $\|P_C(x)-z\|^2 \leq \|x-z\|^2-\|x-P_C(x)\|^2, \forall z \in C.$

Definition 2.1. A mapping $F : H \to H$ is said to be

(i) L-Lipschitz continuous on H if

$$\|F(x) - F(y)\| \leq L\|x - y\|, \forall x, y \in H;$$

(ii) monotone on H if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \forall x, y \in H;$$

(iii) pseudomonotone on H if

$$\langle F(x), y - x \rangle \geq 0 \Rightarrow \langle F(y), y - x \rangle \geq 0, \forall x, y \in H.$$

The following crucial fact remains valid for pseudomonotone mappings.

Lemma 2.2. *[8] Let $G : H \to H$ be pseudomonotone and L-Lipschitz continuous on H such that $Sol(C, G) \neq \emptyset$. Lt $x \in H, \lambda > 0$ and define $y = P_C(x - \lambda G(x)), z = P_T(x - \lambda G(y))$, where*

$$T := \{w \in H : \langle x - \lambda G(x) - y, w - y \rangle \leq 0\}.$$

Then for all $x^* \in Sol(C, G)$, we have

$$\|z-x^*|^2 \leq \|x-x^*\|^2-(1-\lambda L)\|x-y\|^2-(1-\lambda L)\|y-z\|^2.$$

Lemma 2.3. *[9] Let $\{s_n\}$ be a sequence of nonnegative numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ and $\{c_n\}$ be a sequence of real numbers satisfying the conditions:*

(i) $s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n c_n,$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty, \limsup_{n\to\infty} c_n \leq 0.$

Then $\lim_{n\to\infty} s_n = 0.$

Lemma 2.4. *[10] Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \leq a_{n_0+1}$. For all integer $n \geq n_0$, define*

$$\tau(n) = max\{k \in \mathbb{N} : n_0 \leq k \leq n, a_k \leq a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a non-decreasing sequence satisfying $lim_{n\to\infty}\tau(n) = \infty$ and the following inequalities hold true:

$$a_{\tau(n)} \leq a_{\tau(n)+1}, \; a_n \leq a_{\tau(n)+1} \quad \forall n \geq n_0.$$

Lemma 2.5. *[8] Suppose that $G : C \to H$ is pseudomonotone on C and $limsup_{k\to\infty}\langle G(x^k), y\rangle \leq \langle G(\bar{x}), y\rangle$ for every sequence $\{x^k\} \subset C$ converging weakly to $\bar{x}$ and $y \in H$. Then, the solution set of* VIP(C, G) *is closed and convex.*

3 THE ALGORITHM AND CONVERGENCE ANALYSIS

In what follows, we impose the following assumptions on the mappings F_1, F_2 associated with the split pseudomonotone variational inequality.

(A1): $F_1 : H_1 \to H_1$ is pseudomonotone and L_1-Lipschitz continuous.

(A2): $\limsup_{k\to\infty}\langle F_1(x^k), y - y^k\rangle \le \langle F_1(\bar{x}, y - \bar{y})\rangle$ for every sequences $\{x^k\}, \{y^k\}$ converging weakly to $\bar{x}$ and $\bar{y}$, respectively.

(A3): $F_2 : H_2 \to H_2$ is pseudomonotone and L_2-Lipschitz continuous.

(A4): $\limsup_{k\to\infty}\langle F_2(u^k), v - v^k\rangle \le \langle F_2(\bar{u}, v - \bar{v})\rangle$ for every sequences $\{u^k\}, \{v^k\}$ converging weakly to $\bar{u}$ and $\bar{v}$, respectively.

Let have some remarks on the above assumptions.

(i) Assumptions (A1)-(A4) are widely used in the theory of VIPs.

(ii) In finite dimension spaces, condtions (A2) and (A4) automatically follow from the Lipschitz continuity of F_1, F_2.

(iii) If F_1 and F_2 satisfy the properties (A1) and (A3), respectively, then by Lemma 2.5, the solution sets $\mathrm{Sol}(C, F_1)$ and $\mathrm{Sol}(Q, F_2)$ of the variational inequalities $\mathrm{VIP}(C, F_1)$ and $\mathrm{VIP}(Q, F_2)$ are closed and convex. For the sake of convenience, an empty set is considered to be closed and convex. Therefore, the solution set $\Omega = \{x^* \in \mathrm{Sol}(C, F_1) : Bx^* \in \mathrm{Sol}(Q, F_2)$ of the (SVIP) is also closed and convex.

The algorithm presented as follows is for finding the x^0-minimum-norm solution of a split pseudomonotone variational inequality (SVIP) in Hilbert spaces.

Algorithm 1. Choose $x^0 \in H_1$, the sequences $\{\alpha_k\} \subset (0,1), \{\delta_k\}, \{\lambda_k\}$ and $\{\mu_k\}$ such that

$$\begin{cases} \lim_{k\to\infty}\alpha_k = 0, \ \sum_{k=0}^{\infty}\alpha_k = \infty, \\ \{\delta_k\} \subset [a,b] \quad \text{for some} \quad a, b \in (0, \frac{1}{\|A\|^2+1}), \\ \{\lambda_k\} \subset [c,d] \quad \text{for some} \quad c, d \in (0, \frac{1}{L_1}), \\ \{\mu_k\} \subset [e,f] \quad \text{for some} \quad e, f \in (0, \frac{1}{L_2}). \end{cases}.$$

For each iteration $k \ge 0$, compute

$$u^k = Ax^k,$$
$$v^k = P_Q(u^k - \mu_k F_2(u^k)),$$
$$w^k = P_{Q_k}(u^k - \mu_k F_2(v^k)),$$

where $Q_k := \{w_2 \in H_2 : \langle u^k - \mu_k F_2(u^k) - v^k, w_2 - v^k\rangle \le 0\}$. Further, we compute

$$y^k = x^k + \delta_k B^*(w^k - u^k),$$
$$t^k = P_C(y^k - \lambda_k F_1(y^k)),$$
$$z^k = P_{C_k}(y^k - \lambda_k F_1(t^k)),$$

where $C_k := \{w_1 \in H_1 : \langle y^k - \lambda_k F_1(y^k) - t^k, w_1 - t^k\rangle \le 0\}$, and define the next iteration as

$$x^{k+1} = \alpha_k x^0 + (1-\alpha_k)z^k \quad k \ge 0.$$

Lemma 3.1. *Suppose that the assumptions* (A1)-(A4) *and* $\Omega \neq \emptyset$ *hold. Then, the sequences* $\{x^k\}, \{y^k\}$ *and* $\{z^k\}$ *generated by Algorithm 1 satisfy the following inequality*

$$\|z^k - x^*\| \le \|y^k - x^*\| \le \|x^k - x^*\| \quad \forall k,$$

where x^* *is a unique solution of the split variational inequality* (SVIP) (1) - (2) *such that* $x^* = P_\Omega(x^0)$.

Chứng minh. Since $\Omega \neq \emptyset$, problem (SVIP) (1) - (2) has a unique solution, denoted by x^*. From Lemma 2.2, we have for all k,

$$\|z^k - x^*\|^2 \le \|y^k - x^*\|^2 - (1-\lambda_k L_1)\|y^k - t^k\|^2 - (1-\lambda_k L_1)\|t^k - z^k\|^2, \tag{4}$$

$$\|w^k - B^*\|^2 \le \|u^k - Bx^*\|^2 - (1-\mu_k L_2)\|u^k - v^k\|^2 - (1-\mu_k L_2)\|v^k - w^k\|^2. \tag{5}$$

From (4), (5) and the conditions $\{\lambda_k\} \subset [c,d] \subset (0, \frac{1}{L_1})$ and $\{\mu_k\} \subset [e,f] \subset (0, \frac{1}{L_2})$, we get

$$\|z^k - x^*\| \le \|y^k - x^*\| \quad \forall k, \tag{6}$$

$$\|w^k - Bx^*\| \le \|u^k - Bx^*\| \quad \forall k. \tag{7}$$

From the last inequality (7) and $u^k = Bx^*$, we have that for all k

$$\begin{aligned} &\|y^k - x^*\|^2 \\ &= \|x^k - \delta_k B^*(w^k - u^k) - x^*\|^2 \\ &= \|x^k - x^*\|^2 + \|\delta_k B^*(w^k - u^k)\|^2 \\ &+ 2\delta_k\langle x^k - x^*, B^*(w^k - u^k)\rangle \\ &\le \|x^k - x^*\|^2 + \delta_k^2\|B^*\|^2\|w^k - u^k\|^2 \\ &+ 2\delta_k\langle B(x^k - x^*), w^k - u^k\rangle \\ &= \|x^k - x^*\|^2 + \delta_k^2\|B\|^2\|w^k - u^k\|^2 \\ &+ 2\delta_k[\langle w^k - Bx^*, w^k - u^k\rangle - \|w^k - u^k\|^2] \\ &\le \|x^k - x^*\|^2 + \delta_k^2\|B\|^2\|w^k - u^k\|^2 \\ &- \delta_k\|w^k - u^k\|^2 \\ &= \|x^k - x^*\|^2 - \delta_k(1 - \delta_k\|B\|^2)\|w^k - u^k\|^2. \end{aligned} \tag{8}$$

Due to $\delta_k \in [a,b] \subset \left(0, \frac{1}{\|B\|^2+1}\right)$, (5) and (8) we obtain

$$\|z^k - x^*\| \le \|y^k - x^*\| \le \|x^k - x^*\| \quad \forall k.$$

This completes the proof. $\square$

Theorem 3.1. *Suppose that the assumptions* (A1)-(A4) *and* $\Omega \neq \emptyset$ *hold. Then, the sequence* $\{x^k\}$ *generated by Algorithm 1 converges strongly to* $x^* \in \Omega$*, the solution set of* (SVIP) (1)-(2) *satisfying* $x^* = P_\Omega(x^0)$.

Chứng minh. We will divide the proof into several steps as follow.

Step 1. The sequences $\{x^k\}, \{y^k\}$, and $\{z^k\}$ generated by Algorithm 1 are bounded. Based on Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \|(1-\alpha_k)(z^k - x^*) - \alpha_k(x^* - x^0)\| \\ &\le (1-\alpha_k)\|x^k - x^*\| + \alpha_k\|x^* - x^0\| \\ &\le \dots \\ &\le \|x^0 - x^*\|. \end{aligned} \tag{9}$$

Then the sequence $\{x^k\}$ is bounded and so are the sequences $\{y^k\}$ and $\{z^k\}$.

Step 2. For all k we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 \le &(1-\alpha_k)\|x^k - x^*\|^2 \\ &- 2\alpha_k\langle x^* - x^0, x^{k+1} - x^*\rangle,\end{aligned}$$

where x^* is the unique solution of the (SVIP) (1) - (2).

Indeed, using the inequality

$$\|x - y\|^2 \le \|x\|^2 - 2\langle y, x - y\rangle \quad \forall x, y \in H,$$

and from Lemma 2.1 and Lemma 3.1, we obtain that

$$\begin{aligned}&\|x^{k+1} - x^*\|^2 \\ &= \|(1-\alpha_k)(z^k - x^*) - \alpha_k(x^* - x^0)\|^2 \\ &\le (1-\alpha_k)^2\|z^k - x^*\|^2 - 2\alpha_k\langle x^* - x^0, x^{k+1} - x^*\rangle \\ &\le (1-\alpha_k)\|z^k - x^*\|^2 - 2\alpha_k\langle x^* - x^0, x^{k+1} - x^*\rangle \\ &\le (1-\alpha_k)\|x^k - x^*\|^2 - 2\alpha_k\langle x^* - x^0, x^{k+1} - x^*\rangle\end{aligned}$$

Step 3. We show that $\{x^k\}$ converges strongly to the unique solutions x^* of the (SVIP) (1) - (2).

Now, we consider two posible cases:

Case 1. There exist k_0 such that sequence $\{\|x^k - x^*\|\}$ is decreasing for $k \ge k_0$. In this case, there exists the limit of $\{\|x^k - x^*\|\}$.

From Lemma 3.1 and the proof in Step 2, we have

$$\begin{aligned}0 &\le \|y^k - x^*\|^2 - \|z^k - x^*\|^2 \\ &\le \|x^k - x^*\|^2 - \|z^k - x^*\|^2 \\ &\le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \\ &- 2\alpha_k\langle x^* - x^0, x^{k+1} - x^*\rangle\end{aligned} \tag{10}$$

Since the limit of $\{\|x^k - x^*\|\}$ exists, $\lim_{k\to\infty}\alpha_k = 0$, $\{x^k\}$ and $\{z^k\}$ are two bounded sequences, it follows from (10) that

$$\begin{aligned}&\lim_{k\to\infty}\|y^k - x^*\|^2 - \|z^k - x^*\|^2 = 0 \text{ and} \\ &\lim_{k\to\infty}\|x^k - x^*\|^2 - \|z^k - x^*\|^2 = 0.\end{aligned} \tag{11}$$

One can deduce from the last equalities that

$$\lim_{k\to\infty}\|x^k - x^*\|^2 - \|y^k - x^*\|^2 = 0. \tag{12}$$

From (4) and $\{\lambda_k\} \subset [c,d] \subset \left(0, \frac{1}{L_1}\right)$, we obtain

$$(1 - dL_1)\|y^k - t^k\|^2 \le \|y^k - x^*\|^2 - \|z^k - x^*\|^2. \tag{13}$$

Hence, it follows directly from (11) and (13) that

$$\lim_{k\to\infty}\|y^k - t^k\| = 0. \tag{14}$$

Now, from (8) and $\{\delta_k\} \subset [a,b] \subset \left(0, \frac{1}{\|B\|^2+1}\right)$, we have

$$a(1 - b\|B\|^2)\|w^k - u^k\|^2 \le \|x^k - x^*\|^2 - \|y^k - x^*\|^2.$$

Combine the last inequality and (13) we get

$$\lim_{k\to\infty}\|w^k - u^k\| = 0.$$

Note that, for all k, we have

$$\begin{aligned}\|y^k - x^k\| = \|\delta_k B^*(w^k - u^k)\| &\le \delta_k\|B^*\|\|w^k - u^k\| \\ &\le b\|B\|\|w^k - u^k\|\end{aligned}$$

Using the last inequality together with $\lim_{k\to\infty}\|w^k - u^k\| = 0$, we can deduce that

$$\lim_{k\to\infty}\|y^k - x^k\| = 0. \tag{15}$$

From (14) and (15) we have

$$\lim_{k\to\infty}\|x^k - t^k\| = 0. \tag{16}$$

We prove that $\liminf_{k\to\infty}\langle x^* - x^0, x^{k+1} - x^*\rangle \ge 0$. Choose a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that

$$\begin{aligned}&\liminf_{k\to\infty}\langle x^* - x^0, x^{k+1} - x^*\rangle \\ &= \liminf_{k\to\infty}\langle x^* - x^0, x^{k_n} - x^*\rangle.\end{aligned}$$

Due to the boundedness of $\{x^{k_n}\}$, we may assume that $x^{k_n} \rightharpoonup \bar{x} \in H_1$. Therefore, we can write

$$\liminf_{k\to\infty}\langle x^* - x^0, x^{k+1} - x^*\rangle = \langle x^* - x^0, \bar{x} - x^*\rangle. \tag{17}$$

From (15) and (16) and $x^{k_n} \rightharpoonup \bar{x}$ we also deduce that $y^{k_n} \rightharpoonup \bar{x}$ and $t^{k_n} \rightharpoonup \bar{x}$ as $n \to \infty$. Since $\{t^{k_n}\} \subset C$ and C is weakly closed then $\bar{x} \in C$. From (16), it follows that $\{x^k - t^k\}$ is a bounded sequence. Since $\{x^k\}$ is bounded so is $\{t^k\}$.

Now, we show that $\bar{x} \in \operatorname{Sol}(C, F_1)$. Indeed, from the definition of $\{t^{k_n}\}$ we have

$$\langle y^{k_n} - \lambda_{k_n} F_1(y^{k_n}) - t^{k_n}, x - t^{k_n}\rangle \le 0 \quad \forall n.$$

Since $\lambda_{k_n} \ge 0$ for every n, it follows from the above inequality that

$$\langle F_1(y^{k_n}), x - t^{k_n}\rangle \ge \frac{\langle y^{k_n} - t^{k_n}, x - t^{k_n}\rangle}{\lambda_{k_n}}. \quad (18)$$

Using Cauchy – Schwarz inequality and the fact that $\lambda_{k_n} \ge c > 0$ for all n we get

$$\left|\frac{\langle y^{k_n} - t^{k_n}, x - t^{k_n}\rangle}{\lambda_{k_n}}\right| \le \frac{\|y^{k_n} - t^{k_n}\|\|x - t^{k_n}\|}{c}. \quad (19)$$

Since $\|y^{k_n} - t^{k_n}\| \to 0$ as $n \to \infty$ and sequence $\{t^{k_n}\}$ is bounded, we have

$$\lim_{n\to\infty} \frac{\|y^{k_n} - t^{k_n}\|\|x - t^{k_n}\|}{c} = 0.$$

The last limit together with (19) ensure that $\lim_{n\to\infty} \frac{\langle y^{k_n} - t^{k_n}, x - t^{k_n}\rangle}{\lambda_{k_n}} = 0$. So, using (18), condition (A2) and the weak convergence of two sequences $\{y^{k_n}\}$ and $\{t^{k_n}\}$ to $\bar{x}$, we obtain

$$0 \le \operatorname{limsup}_{n\to\infty} \langle F_1(y^{k_n}), x - t^{k_n}\rangle \le \langle F_1(\bar{x}), x - \bar{x}\rangle,$$

which means $\bar{x} \in \operatorname{Sol}(C, F_1)$. Since $\{x^k\}$ is bounded, then $\{u^k = Bx^k\}$ is bounded. This together with $\lim_{k\to\infty}\|w^k - u^k\| = 0$ implies that $\{w^k\}$ is bounded.

Now, from (7) and the triangle inequality, we get, for all k,

$$\begin{aligned} 0 &\le \|u^k - Bx^*\|^2 - \|w^k - Bx^*\|^2 \\ &\le (\|u^k - Bx^*\| + \|w^k - Bx^*\|)\|u^k - w^k\|. \end{aligned}$$

The last evaluation, $\lim\|w^k - u^k\| = 0$ and the boundedness of two sequences $\{u^k\}$ and $\{w^k\}$, we gain

$$\lim_{k\to\infty}(\|u^k - Bx^*\|^2 - \|w^k - Bx^*\|^2) = 0. \quad (20)$$

From (5) and $\{\mu_k\} \subset [e, f] \subset \left(0, \frac{1}{L_2}\right)$, we have

$$(1 - fL_2)\|u^k - v^k\|^2 \le \|u^k - Bx^*\|^2 - \|w^k - Bx^*\|^2.$$

Hence, together with (20) one can deduce that

$$\lim_{k\to\infty}(\|u^k - v^k\|^2 = 0. \quad (21)$$

The boundedness of $\{u^k\}$ and (21) ensure that $\{v^k\}$ is bounded.

From $x^{k_n} \rightharpoonup \bar{x}$, we get $u^{k_n} = Bx^{k_n} \rightharpoonup B\bar{x}$. From (21), we have $v^{k_n} \rightharpoonup B\bar{x}$. Since $\{v^{k_n}\} \subset Q$ and Q is closed and convex, is it also weakly closed, and thus, $B\bar{x} \in Q$.

Next, we prove that $B\bar{x} \in \operatorname{Sol}(Q, F_2)$.

Indeed, let $y \in Q$. Since $v^{k_n} = P_Q(u^{k_n} - \mu_{k_n} F_2(u^{k_n}))$ then

$$\langle u^{k_n} - \mu_{k_n} F_2(u^{k_n}) - v^{k_n}, y - v^{k_n}\rangle \le 0 \quad \forall n.$$

Since $\mu_{k_n} > 0$ for every n, it follows from the last inequality that

$$\langle F_2(u^{k_n}), y - v^{k_n}\rangle \ge \frac{\langle u^{k_n} - v^{k_n}, y - v^{k_n}\rangle}{\mu_{k_n}}. \quad (22)$$

Using Cauchy – Schwarz inequality and the fact that $\mu_{k_n} \ge e > 0$ for all n, we find this inequality

$$\left|\frac{\langle u^{k_n} - v^{k_n}, y - v^{k_n}\rangle}{\mu_{k_n}}\right| \le \frac{\|u^{k_n} - v^{k_n}\|\|y - v^{k_n}\|}{e}. \quad (23)$$

Since $\|u^{k_n} - v^{k_n}\| \to 0$ as $n \to \infty$ and sequence $\{v^{k_n}\}$ is bounded, it follows that

$$\lim_{n\to\infty} \frac{\|u^{k_n} - v^{k_n}\|\|y - v^{k_n}\|}{e} = 0.$$

Using (19) and the last fact, we get $\lim_{n\to\infty} \frac{\langle u^{k_n} - v^{k_n}, y - v^{k_n}\rangle}{\mu_{k_n}} = 0$. So, using (22), condition (A4) and the weak convergence of two sequences $\{u^{k_n}\}$ and $\{v^{k_n}\}$ to $B\bar{x}$, we attain

$$\begin{aligned} 0 &\le \operatorname{limsup}_{n\to\infty} \langle F_2(u^{k_n}), y - v^{k_n}\rangle \\ &\le \langle F_2(B\bar{x}), y - B\bar{x}\rangle, \end{aligned}$$

which means $B\bar{x} \in \operatorname{Sol}(Q, F_2)$.

Thus, $\bar{x} \in \Omega$. As $x^* \in \Omega$, $\bar{x} \in \Omega$, and $x^* = P_\Omega(x^0)$, it follows that $\langle x^0 - x^*, \bar{x} - x^*\rangle \ge 0$. So, from (17), we gain $\operatorname{liminf}_{k\to\infty}\langle x^* - x^0, x^{k+1} - x^*\rangle \ge 0$.

From the results obtained in Step 2, we get

$$\|x^{k+1} - x^*\|^2 \le (1 - \alpha_k)\|x^k - x^*\|^2 + \alpha_k c_k,$$

where

$$c_k = \langle x^* - x^0, x^{k+1} - x^*\rangle.$$

Using the fact $\operatorname{limsup}_{k\to\infty} c_k \le 0$.

By Lemma 2.2, we have $\lim_{k\to\infty}\|x^k - x^*\|^2 = 0$, that is $x^k \to x^*$ as $k \to \infty$.

Case 2. Suppose that there exists a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that

$$\|x^{k_n} - x^*\| \le \|x^{k_n+1} - x^*\| \quad \forall n \in \mathbb{N}.$$

According to Lemma 2.4, there exists a nondecreasing sequence $\{\tau(k)\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}$ ad the following inequalities hold true for all sufficiently large $n \in \mathbb{N}$.

$$\begin{aligned} &\|x^{\tau(k)} - x^*\| \le \|x^{\tau(k)+1} - x^*\|, \\ &\|x^k - x^*\| \le \|x^{\tau(k)+1} - x^*\|. \end{aligned} \tag{24}$$

Combine (24) and (9), we get

$$\begin{aligned} &\|x^{\tau(k)} - x^*\| \le \|x^{\tau(k)+1} - x^*\| \qquad (25) \\ &\le (1-\alpha_{\tau(k)})\|z^{\tau(k)} - x^*\|^2 + \alpha_{\tau(k)}\|x^* - x^0\| \end{aligned}$$

Also, from Lemma 3.1 and (25), we get that

$$\begin{aligned} 0 &\le \|y^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| \\ &\le \|x^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| \\ &\le -\alpha_{\tau(k)}\|z^{\tau(k)} - x^*\| + \alpha_{\tau(k)}\|x^* - x^0\| \end{aligned} \tag{26}$$

Since, $\lim_{k\to\infty}\alpha_k = 0$ abd $\{z^k\}$ is bounded, together with (26), one can deduce that

$$\begin{aligned} &\lim_{k\to\infty}\|y^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| = 0 \\ &\lim_{k\to\infty}\|x^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| = 0 \end{aligned} \tag{27}$$

Now, from (26), we attain

$$\lim_{k\to\infty}\|x^{\tau(k)} - x^*\| - \|y^{\tau(k)} - x^*\| = 0 \tag{28}$$

Hence, from (27) and (28) and the boundedness of sequences $\{x^k\}, \{y^k\}, \{z^k\}$, we can write that

$$\lim_{k\to\infty}\left(\|y^{\tau(k)} - x^*\|^2 - \|z^{\tau(k)} - x^*\|^2\right) = 0, \tag{29}$$

$$\lim_{k\to\infty}\left(\|x^{\tau(k)} - x^*\|^2 - \|y^{\tau(k)} - x^*\|^2\right) = 0. \tag{30}$$

From (4) and $\{\lambda_k\} \subset [c,d] \subset \left(0, \frac{1}{L_1}\right)$, we have

$$\begin{aligned} &(1-dL_1)\|y^{\tau(k)} - t^{\tau(k)}\|^2 + (1-dL_1)\|t^{\tau(k)} - z^{\tau(k)}\|^2 \\ &\le \|y^{\tau(k)} - x^*\|^2 - \|z^{\tau(k)} - x^*\|^2. \end{aligned}$$

Thus, it follows from (29) that

$$\begin{aligned} &\lim_{k\to\infty}\|y^{\tau(k)} - t^{\tau(k)}\| = 0, \\ &\lim_{k\to\infty}\|t^{\tau(k)} - z^{\tau(k)}\| = 0. \end{aligned} \tag{31}$$

Now, from (8) and $\{\delta_k\} \subset [a,b] \subset \left(0, \frac{1}{\|B\|^2+1}\right)$, we have

$$\begin{aligned} &a(1-b\|B\|^2)\|w^{\tau(k)} - u^{\tau(k)}\|^2 \\ &\le \|x^{\tau(k)} - x^*\|^2 - \|y^{\tau(k)} - x^*\|^2. \end{aligned}$$

From the last inequality and (30) we get

$$\lim_{k\to\infty}\|w^{\tau(k)} - u^{\tau(k)}\| = 0. \tag{32}$$

We also have

$$\begin{aligned} \lim_{k\to\infty}\|y^{\tau(k)} - x^{\tau(k)}\| &= \|\delta_{\tau(k)}B^*(w^{\tau(k)} - u^{\tau(k)})\| \\ &\le \delta_{\tau(k)}\|B^*\|\|w^{\tau(k)} - u^{\tau(k)}\| \\ &\le b\|B\|\|w^{\tau(k)} - u^{\tau(k)}\| \end{aligned}$$

Combining the above inequality with (32), we get

$$\lim_{k\to\infty}\|y^{\tau(k)} - x^{\tau(k)}\| = 0. \tag{33}$$

From (31), (32) and the triangle inequality, we have arguing similarly as in the first case, we can conclude that

$$\liminf_{k\to\infty}\langle x^* - x^0, x^{\tau(k)} - x^*\rangle = 0. \tag{34}$$

Now, consider

$$\begin{aligned} \|x^{\tau(k)+1} - x^{\tau(k)}\| &= \|(1-\alpha_{\tau(k)})(z^{\tau(k)} - x^{\tau(k)}) \\ &\qquad + \alpha_{\tau(k)}(x^0 - x^{\tau(k)})\| \\ &\le \|z^{\tau(k)} - x^{\tau(k)}\| + \alpha_{\tau(k)}\|x^0 - x^{\tau(k)})\|. \end{aligned} \tag{35}$$

From $\lim_{k\to\infty}\alpha_k = 0$, the boundedness of $\{x^k - x^0\|\}$, (32) and (35), we gain

$$\lim_{k\to\infty}\|x^{\tau(k)+1} - x^{\tau(k)}\| = 0. \tag{36}$$

From (36) and Cauchy – Schwarz inequality, we obtain that

$$\begin{aligned} &|\langle x^* - x^0, x^{\tau(k)+1} - x^{\tau(k)}\rangle| \\ &\le \|x^* - x^0\|\|x^{\tau(k)+1} - x^{\tau(k)}\| \end{aligned}$$

which means that

$$\lim_{k\to\infty}\langle x^* - x^0, x^{\tau(k)+1} - x^{\tau(k)}\rangle = 0. \tag{37}$$

One can deduce from (34) and (37) that

$$\begin{aligned} &\liminf_{k\to\infty}\langle x^* - x^0, x^{\tau(k)+1} - x^{\tau(k)}\rangle \\ &= \liminf_{k\to\infty}[\langle x^* - x^0, x^{\tau(k)} - x^*\rangle \\ &\qquad - \langle x^* - x^0, x^{\tau(k)+1} - x^{\tau(k)}\rangle] \\ &= \liminf_{k\to\infty}\langle x^* - x^0, x^{\tau(k)} - x^{\tau(k)}\rangle \\ &\ge 0. \end{aligned} \tag{38}$$

From the proof in Step 2 and (25), it leads to

$$\begin{aligned} |\|x^{\tau(k)+1} - x^*\| &\le \|(1-\alpha_{\tau(k)})\|x^{\tau(k)} - x^*\|^2 \\ &\quad - 2\alpha_{\tau(k)}\langle x^* - x^0, x^{\tau(k)+1} - x^*\rangle, \\ &\le \|(1-\alpha_{\tau(k)})\|x^{\tau(k)} - x^*\| \\ &\quad - 2\alpha_{\tau(k)}\langle x^* - x^0, x^{\tau(k)+1} - x^*\rangle, \end{aligned}$$

and again, from (26) and $\alpha_{\tau(k)} > 0$, we obtain

$$\begin{aligned} \|x^k - x^*\|^2 &\le \|x^{\tau(k)+1} - x^*\|^2 \\ &\le -2\langle x^* - x^0, x^{\tau(k)+1} - x^*\rangle. \end{aligned} \tag{39}$$

Taking the limit in (39) as $k \to \infty$ and using (38), we deduce that

$$\limsup_{k\to\infty}\|x^k - x^*\|^2 \le 0$$

which means $x^k \to x^*$ as $k \to \infty$. This completes the proof of Theorem 1. □

When $F_1 = 0$ and $F_2 = 0$ we have the following algorithm as a special case of Algorithm 1 for solving a split feasibility problem in Hilbert space.

Algorithm 2. Choose $x^0 \in H_1$ and the sequences $\{\alpha_k\}$ and $\{\delta_k\}$ satisfying the following conditions

$$\begin{cases} \{\alpha_k\} \subset (0,1),\ \lim_{k\to\infty}\alpha_k = 0,\ \sum_{k=0}^{\infty}\alpha_k = \infty, \\ \{\delta_k\} \subset [a,b] \subset \left(0, \frac{1}{\|B\|^2+1}\right). \end{cases}$$

For each iteration $k \geq 0$, compute

$$\begin{cases} y^k = (I - P_Q)Bx^k \\ x^{k+1} = \alpha_k x^0 + (1-\alpha_k)P_C(x^k - \delta_k B^* y^k). \end{cases} \tag{40}$$

Corollary 3.1. *Let C and Q be the two closed and convex subsets of two Hilbert spaces H_1 and H_2, respectively and let $B : H_1 \to H_2$ be a bounded linear operator. Let $\{x^k\}$ be the sequence generated by Algorithm 2. Then $\{x^k\}$ converges strongly to the unique solution x^* of the split feasibility problem (SFP) such that $x^* = P_\Psi(x^0)$, provided that the solution set $\Psi = \{x^* \in C : Bx^* \in Q\}$ of the (SFP) is nonempty.*

4 NUMERICAL RESULTS

Let $H_1 = H_2 = \mathbb{R}^4$ and let $B : \mathbb{R}^4 \to \mathbb{R}^4$ be a bounded linear operator defined by

$$Bx = (x_1 + 2x_2, x_2 + x_4, x_1 - 2x_4, -x_3 + x_4)$$
for each $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Choose C, Q be two subsets of $\mathbb{R}^4$ as follows

$$C := \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 - x_3 - x_4 = 0\}$$
$$Q := \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : -2y_1 + 2y_2 - y_3 - 3y_4 = 0\}$$

Let F_1 and F_2 be two mappings from $\mathbb{R}^4$ to itself defined as follows

$$F_1(x) = (x_1, x_2, x_3, x_4),$$
$$F_2(x) = (x_1, x_2, x_3, x_4).$$

The the solution set $\Omega = \{x^* \in \mathrm{Sol}(C, F_1) : Bx^* \in \mathrm{Sol}(Q, F_2)\}$ is deduced to the problem of finding x^* such that $\|x^*\| = \min\{\|x\| : x \in C\}$ and $y^* = Bx^*$ while $\|y^*\| = \min\{\|y\| : y \in Q\}$. It is easy to find the solution x^* which is $x^* = (0,0,0,0)^T$.

Now, using the Algorithm 1 with $\alpha_k = \frac{1}{k+3}, \mu_k = \delta_k = \lambda_k = 0.01$ satisfying the conditions of Algorithm 1 and Theorem 3.1 and $x^0 = (1,3,1,1)^T \in \mathbb{R}^4$, the stopping rule of the iteration is $\|x^k - x^{k-1}\| \leq$ err. The following table shows the approximate solutions $x_n = (x_1^k, x_2^k, x_3^k, x_4^k) \in \mathbb{R}^4$ of the above problem with the corresponding parameters.

The experimental results shown in the above table illustrate the convergence of the considered iterative method for solving a split variational inequality.

5 CONCLUSION

We have presented in this paper an subgradient extragradient method for solving a split pseudomonotone variational inequality in Hilbert spaces and a special case of the proposed one for solving a split feasibility problem. The strong convergence of the methods is proven under some certain assumptions imposed on the mappings F_1 and F_2 of the split variational inequality which is the pseudomonotocity and a numerical example for the convergence illustration of the proposed method is given.

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