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MODIFIED PROJECTION ALGORITHMS FOR STRONGLY PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract:

The variational inequality problem have many important applications in the fields of signal processing, image processing, optimal control and many others. In this paper, we introduce two projection algorithms for solving strongly pseudomonotone variational inequalities. The considered methods are based on some existing ones. Our algorithms use dynamic step-sizes, chosen based on information of previous steps and their strong convergence is proved without the Lipschitz continuity of the underlying mappings. Some numerical experiments are presented to verify the effectiveness of the proposed algorithms.

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PHƯƠNG PHÁP CHIẾU GIẢI BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN GIẢ ĐƠN ĐIỆU MẠNH

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Thông tin bài viết

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Từ khóa:
Bài toán bất đẳng thức biến phân, không gian Hilbert, giả đơn điệu mạnh, độ phức tạp của thuật toán.

Tóm tắt:

Bài toán bất đẳng thức biến phân có nhiều ứng dụng quan trọng trong các lĩnh vực xử lý tín hiệu, xử lý ảnh, điều khiển tối ưu và nhiều ứng dụng. Trong bài báo này, chúng tôi giới thiệu hai thuật toán để giải các bất đẳng thức biến phân giả đơn điệu mạnh. Phương pháp mới cải thiện một số thuật toán hiện có. Các thuật toán của chúng tôi sử dụng cỡ bước tự thích nghi, được xây dựng dựa trên thông tin của bước trước và sự hội tụ mạnh của các phương pháp này được chứng minh mà không cần tính liên tục Lipschitz của các ánh xạ giá. Chúng tôi tiến hành một vài thử nghiệm số để minh họa tính hiệu quả của các thuật toán mới.

1 Introduction

Let C be a nonempty, closed and convex set in Hilbert space H, $F : C \to C$ be a mapping. The variational inequality problem of F on C is

$$\text{find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0 \, \forall y \in C. \quad (\text{VIP}(F, C))$$

This problem is an important tool in economics, operations research, and mathematical physics. It includes many problems of nonlinear analysis in a unified form, such as optimization, fixed point problems, Nash equilibrium problems, saddle point problems.

The simplest iterative procedure for a variational inequality problem in a Hilbert space H may be well-known projected gradient method

$$\begin{cases} x^0 \in C \\ x^{k+1} = P_C\left(x^k - \lambda_k F(x^k)\right) \end{cases} \quad (1.1)$$

Under the assumptions that F is γ-strongly pseudomonotone and L-Lipschitz continuous on C, $\lambda \in (0, 2\frac{\gamma}{L^2})$, the sequence $\{x^k\}$ generated by (1.1) converges linearly to the unique solution of the problem (VIP(F, C)).

If the Lipschitz continuity of F is eliminated and $\{\lambda_k\}$ is bounded away from zero, algorithm (1.1), in general, is not convergent. In this case, we need to use step sizes tending to zero. In 2010, Bello Cruz et al. [6] proposed the following self-adaptive algorithm

$$\begin{cases} x^0 \in C \\ \lambda_k = \frac{\beta_k}{\max\{1; \|F(x^k)\|\}} \\ x^{k+1} = P_C\left(x^k - \lambda_k F(x^k)\right), \end{cases} \tag{1.2}$$

where C is a subset of $\mathbb{R}^n$ and $\{\beta_k\}$ is a sequence of nonegative numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k = \infty; \quad \sum_{k=0}^{\infty} \beta_k^2 < \infty.$$

Under the assumption that F is paramonotone, the authors proved that the sequence $\{x^k\}$ generated by (1.2) converges to a solution of VIP(F, C). However, the condition $\sum_{i=0}^{\infty} \beta_i^2 < \infty$, makes the step size of (1.2) tend to zero very fast, and hence, slows down the convergence rate of this algorithm. Moreover, in (1.2), one need $\|F(x^k)\|$. This procedure increases the computational cost of the algorithm. Motivated by the works in [6, 11], in this paper, we introduce two new algorithms for solving (VIP(F, C)). Our algorithms are designed to inherit the advantages and overcome the disadvantages of the existing ones. Namely, in each iteration of the first algorithm, we do not need to compute $\|F(x^k)\|$, and in the second algorithm, we can estimate the maximum iterations to get a given accuracy. Also, the new algorithms do not require the Lipchitz continuity of the involving mapping. Moreover, the steps size λ_k in the new algorithms needs not to satisfy the condition $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$. All these features help to reduce the computational cost and speed up our algorithms.
The remaining part of this paper is organized as follows: the next section presents some notations, definitions and lemmas that will be used in the sequel. The third section is devoted to the proof of our main result. In Section 4, some numerical examples are also given to illustrate the convergence of the proposed algorithms.

2 Preliminaries

We present some notations and preliminary results, which will be used in thenext sections. We refer the reader to [5, 22] for more details.
For each $x \in H$, denote

$$P_C(x) := \text{argmin}\{\|z - x\| : z \in C\}.$$

The mapping $P_C : x \mapsto P_C(x)$ is called the projection onto C.

Proposition 2.1. [5] *For all $x, y \in H$, it holds that:*

(i) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$,

(ii) $\langle y - P_C(x), x - P_C(x)\rangle \leq 0$.

Definition 2.1. A mapping $F : C \to H$ is called

1. monotone on C if for all $x, y \in C$,

$$\langle F(x) - F(y), x - y\rangle \geq 0;$$

2. γ-strongly monotone on C if there exists a constant $\gamma \in (0, \infty)$ such that for all $x, y \in C$,

$$\langle F(x) - F(y), x - y\rangle \geq \gamma\|x - y\|^2;$$

3. γ-strongly pseudomonotone on C if there exists a constant $\gamma \in (0, \infty)$ such that for all $x, y \in C$,

$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq \gamma\|x - y\|^2.$$

3 Main Results

In this paper, we consider the problem VIP(F, C) under the following conditions:

Assumption 3.1.

(C1) The mapping F is γ-strongly pseudomonotone on C.

(C2) The mapping F is bounded on bounded subsets of C.

(C3) The solution set of VIP(F, C) is not empty.

Under these conditions, the problem VIP(F, C) has a unique solution x^*. In order to find this solution, we propose the following algorithm:

Algorithm 3.1.
Step 0. Choose $x^0 \in C$ and a nonincreasing sequence $\{\lambda_k\} \subset (0, \infty)$ satisfying $\lambda_k \to 0$, $\sum_{i=0}^{\infty} \lambda_k = \infty$. Set $k = 0$.
If C is bounded **then** $K = C$ **else**
$K = C \cap \{x \in R^n : \gamma\|x - x^0\|^2 \leq \langle F(x^0), x^0 - x\rangle\}$.
Step 1. Given x^k, compute x^{k+1} as follows

$$x^{k+1} = P_K(x^k - \lambda_k F(x^k)).$$

Step 2. If $x^k = x^{k+1}$, then STOP, otherwise update $k := k + 1$ and GOTO Step 1.

As we can see, in Algorithm 3.1, we do not need to calculate any $\|F(x^k)$.

If Algorithm 3.1 stops at step k, using Proposition 2.1-ii, we obtain that x^k is the solution of $\mathrm{VIP}(F, C)$. Consider the case when Algorithm 3.1 does not stop after finite iterations.

Theorem 3.1. *If the conditions (C1)- (C3) in Assumption 3.1 are satisfied. Then, the sequence $\{x^k\}$ generated by Algorithm 3.1 strongly converges to the unique solution x^* of* $\mathrm{VIP}(F, C)$.

Proof. For all $x \in K$, we have

$$\left\langle x^{k+1} - x^k + \lambda_k F(x^k), x^{k+1} - x \right\rangle \leq 0.$$

Hence,

$$\left\langle x^{k+1} - x^k, x^{k+1} - x \right\rangle \leq \lambda_k \left\langle F(x^k), x - x^{k+1} \right\rangle. \tag{3.1}$$

Denote by x^* the unique solution of $\mathrm{VIP}(F, C)$. It implies that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 \\ &\quad + 2\left\langle x^{k+1} - x^k, x^{k+1} - x^* \right\rangle \\ &\leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 \\ &\quad + 2\lambda_k \left\langle F(x^k), x^* - x^{k+1} \right\rangle \ \forall k \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Denote

$$I := \left\{ k \in \mathbb{N} : \left\langle F(x^k), x^* - x^{k+1} \right\rangle \geq -\frac{\gamma}{2}\|x^k - x^*\|^2 \right\}.$$

We have two cases:

Case 1: $|I| = \infty$. We have

$$\begin{aligned} \left\langle F(x^i), x^* - x^i \right\rangle &\geq \left\langle F(x^i), x^{i+1} - x^i \right\rangle \\ &\quad - \frac{\gamma}{2}\|x^i - x^*\|^2 \ \forall i \in I. \end{aligned}$$

Because F is strongly pseudomonotone mapping on C and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|F(x^i)\|\|x^i - x^{i+1}\| &\geq \left\langle F(x^i), x^i - x^{i+1} \right\rangle \\ &\geq \left\langle F(x^i), x^i - x^* \right\rangle - \frac{\gamma}{2}\|x^k - x^*\|^2 \\ &\geq \frac{\gamma}{2}\|x^* - x^i\|^2 \ \forall i \in I. \end{aligned} \tag{3.3}$$

We have F is bounded on K-bounded, so we obtain

$$\begin{aligned} \|x^{k+1} - x^k\| &= \|P_K(x^k - \lambda_k F(x^k)) - P_K(x^k)\| \\ &\leq \lambda_k \|F(x^k)\| \\ &\leq M.\lambda_k \quad \forall k \in \mathbb{N}, \end{aligned} \tag{3.4}$$

where $M := \sup\{\|F(x)\| : x \in K\}$.

It follows from (3.3) and (3.4), we have

$$\frac{\gamma}{2}\|x^* - x^i\|^2 \leq \lambda_i \|F(x^i)\|^2 \leq M^2.\lambda_i,$$

or

$$\|x^* - x^i\| \leq \sqrt{\frac{2\lambda_i}{\gamma}}.M, \ \forall i \in I. \tag{3.5}$$

Take $\epsilon > 0$ arbitrarily. Since $\lambda_k \to 0$ and $|I| = \infty$, there exists a number $k_0 \in I$ such that

$$\max\left\{\lambda_k; \sqrt{\frac{2\lambda_k}{\gamma}}\right\} \leq \frac{\epsilon}{2M} \ \forall k \geq k_0.$$

For all $k \geq k_0$, we will show that $\|x^k - x^*\| \leq \epsilon$. Indeed,

- If $k \in I$, from (3.5), we have $\|x^* - x^k\| \leq M.\sqrt{\frac{2\lambda_i}{\gamma}} \leq M\frac{\epsilon}{2M} = \frac{\epsilon}{2} < \epsilon$.

- If $k \notin I$, let $i(k) := \max\{i \in I : i < k\}$, then $k > i(k) \geq k_0$. It follows from (3.2) that

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \ \forall k \notin I.$$

From (3.4) and (3.5), we obtain

$$\begin{aligned} &\|x^k - x^*\| \\ &\leq \|x^{i(k)+1} - x^*\| \\ &\leq \|x^{i(k)+1} - x^{i(k)}\| + \|x^{i(k)} - x^*\| \\ &\leq M.\left(\lambda_{i(k)} + \sqrt{\frac{2\lambda_{i(k)}}{\gamma}}\right) \\ &\leq M\left(\frac{\epsilon}{2M} + \frac{\epsilon}{2M}\right) = \epsilon. \end{aligned}$$

Therefore, we get $x^k \to x^*$. *Case 2:* $|I| < \infty$. Let $m := \max\{i : i \in I\} + 1$. From (3.2), we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq (1 - \lambda_k \gamma)\|x^k - x^*\|^2 \\ &\leq \prod_{i=m}^{k}(1 - \lambda_i \gamma)\|x^m - x^*\|^2 \ \forall k \geq m. \end{aligned}$$

Therefore, the sequence $\{x^k\}$ is bounded. Because

$$\sum_{k=0}^{\infty} \lambda_k = \infty,$$

which implies that $\lim_{k\to\infty} \prod_{i=m}^{k}(1 - \lambda_i \gamma) = 0$, and hence, $x^k \to x^*$. □

Remark 3.1. In Algorithm 3.1, we do not need to know the constant γ of the strong pseudomonotonicity of F. When this constant is known, we can control the accuracy of the algorithm by the number of iterative steps as follows:

Algorithm 3.2.
Step 0. Let $\epsilon > 0$ be the given accuracy. Choose $x^0 \in C$, $r_0 := \frac{1}{\gamma}\|F(x^0)\|$, $k = 0$.
If C is bounded **then** $K = C$ **else**
$K = C \cap \{x \in R^n : \gamma\|x - x^0\|^2 \leq \langle F(x^0), x^0 - x\rangle\}$.
Set $\lambda := \frac{1}{4}\left(\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} - \sqrt{\frac{2}{\gamma}}\right)^2$, where $M := \sup\{\|F(x)\| : x \in K\}$.
Step 1. Given x^k. If $r_k \leq \epsilon$, then STOP, otherwise compute

$$r_{k+1} = r_k\sqrt{1 - \lambda\gamma}$$
$$x^{k+1} = P_K\left(x^k - \lambda F(x^k)\right).$$

Step 2. If $x^k = x^{k+1}$, then STOP, otherwise update $k := k + 1$ and GOTO Step 1.

Theorem 3.2. *If the conditions (C1)- (C3) in Assumption 3.1 are satisfied. Then, Algorithm 3.2 stops after maximum* $\left[2\log_{(1-\lambda\gamma)}\frac{\gamma\epsilon}{\|F(x^0)\|}\right] + 2$ *steps. Moreover, the final output* x^p *of Algorithm 3.2 satisfies* $\|x^p - x^*\| \leq \epsilon$, *where* x^* *is the unique solution of* VIP(F, C).

Proof. If Algorithm 3.2 stops at step p when $x^p = x^{p+1}$ or $r_p \leq \epsilon$. In the first case, x^p is the solution of VIP(F, C), and hence, $\|x^p - x^*\| = 0 < \epsilon$. In other case, we suppose $r_p \leq \epsilon$ for some $p \in \mathbb{N}$, we will prove that $\|x^p - x^*\| \leq \epsilon$. By the same argument that led us to (3.2), we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + 2\lambda\left\langle F(x^k), x^* - x^{k+1}\right\rangle \ \forall k \in \mathbb{N}. \tag{3.6}$$

Denote

$$I := \left\{k \in \mathbb{N} : \left\langle F(x^k), x^* - x^{k+1}\right\rangle \geq -\frac{\gamma}{2}\|x^k - x^*\|^2\right\},$$
$$J := \{k \in \mathbb{N} : k \leq p\}.$$

We have two cases:
Case 1: $I \cap J = \emptyset$. From (3.6), we have

$$\|x^{k+1} - x^*\|^2 \leq (1 - \lambda\gamma)\|x^k - x^*\|^2 \ \forall k = 0, \ldots, p-1.$$

Hence,

$$\|x^p - x^*\|^2 \leq (1 - \lambda\gamma)^p\|x^0 - x^*\|^2. \tag{3.7}$$

On the other hand, since $\left\langle F(x^*), x^0 - x^*\right\rangle \geq 0$, using the strong pseudomonotonicity of F, we have

$$\|F(x^0)\|\|x^0 - x^*\| \geq \left\langle F(x^0), x^0 - x^*\right\rangle \geq \gamma\|x^0 - x^*\|^2.$$

It follows that

$$\|x^0 - x^*\| \leq \frac{1}{\gamma}\|F(x^0)\|. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\begin{aligned}\|x^p - x^*\| &\leq \left(\sqrt{1 - \lambda\gamma}\right)^p\|x^0 - x^*\| \\ &\leq \left(\sqrt{1 - \lambda\gamma}\right)^p\frac{1}{\gamma}\|F(x^0)\| \\ &= \left(\sqrt{1 - \lambda\gamma}\right)^p r_0 \\ &= r_p \leq \epsilon.\end{aligned}$$

Case 2: $I \cap J \neq \emptyset$.

- If $p \in I$, following the same argument that led us to (3.5), we have

$$\|x^p - x^*\| \leq M.\sqrt{\frac{2\lambda}{\gamma}}. \tag{3.9}$$

We have,

$$\begin{aligned}&\frac{1}{4}\left(\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} - \sqrt{\frac{2}{\gamma}}\right)^2 > 0 \\ &\Leftrightarrow \frac{\epsilon}{M} + \frac{1}{\gamma} - \frac{1}{2}\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} > 0 \\ &\Leftrightarrow \epsilon + \frac{M}{\gamma} - M.\frac{1}{2}\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} > 0 \\ &\Leftrightarrow M.\frac{1}{2}\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} - \frac{M}{\gamma} < \epsilon \\ &\Leftrightarrow M.\frac{1}{2}\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} - \frac{1}{2}M\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma}} < \epsilon \\ &\Leftrightarrow M.\sqrt{\frac{2\lambda}{\gamma}} < \epsilon.\end{aligned} \tag{3.10}$$

It follows from (3.9), (3.10), hence

$$\|x^p - x^*\| \leq M.\sqrt{\frac{2\lambda}{\gamma}} < \epsilon.$$

- If $p \notin I$, let $i(p) := \max\{i : i \in I \cap J\}$, we have

$$\begin{aligned}\|x^p - x^*\| &\leq \|x^{i(p)+1} - x^*\| \\ &\leq \|x^{i(p)+1} - x^{i(p)}\| + \|x^{i(p)} - x^*\| \\ &\leq M.\lambda + M.\sqrt{\frac{2\lambda}{\gamma}}.\end{aligned} \tag{3.11}$$

We have,

$$\begin{aligned}M.\lambda + M.\sqrt{\frac{2\lambda}{\gamma}} &= \frac{M}{\gamma} + \epsilon - \frac{1}{2}M\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} + \\ &+ M.\frac{1}{2}\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma} + 4\frac{\epsilon}{M}} - \frac{1}{2}M\sqrt{\frac{2}{\gamma}}\sqrt{\frac{2}{\gamma}} \\ &= \epsilon.\end{aligned} \tag{3.12}$$

From (3.11), (3.12), then

$$\|x^p - x^*\| \leq M.\lambda + M.\sqrt{\frac{2\lambda}{\gamma}} = \epsilon.$$

Now, prove that if $m \geq 2\log_{(1-\lambda\gamma)} \frac{\gamma\epsilon}{\|F(x^0)\|} + 1$ then $r_m \leq \epsilon$.

$$\begin{aligned} r_m &= \sqrt{\prod_{i=0}^{m-1} (1-\lambda\gamma)\frac{1}{\gamma}\|F(x^0)\|} \\ &= (1-\lambda\gamma)^{\frac{m-1}{2}} \frac{1}{\gamma}\|F(x^0)\| \\ &\leq \epsilon. \end{aligned}$$

Because $0 < 1-\lambda\gamma < 1$, so

$$\begin{aligned} &(1-\lambda\gamma)^{\frac{m-1}{2}} \frac{1}{\gamma}\|F(x^0)\| \leq \epsilon \\ \Leftrightarrow\ &(1-\lambda\gamma)^{\frac{m-1}{2}} \leq \frac{\epsilon\gamma}{\|F(x^0)\|} \\ \Leftrightarrow\ &\log_{(1-\lambda\gamma)} (1-\lambda\gamma)^{\frac{m-1}{2}} \leq \log_{(1-\lambda\gamma)} \frac{\epsilon\gamma}{\|F(x^0)\|} \\ \Leftrightarrow\ &\frac{m-1}{2} \geq \log_{(1-\lambda\gamma)} \frac{\epsilon\gamma}{\|F(x^0)\|} \\ \Leftrightarrow\ &m \geq 2\log_{(1-\lambda\gamma)} \frac{\gamma\epsilon}{\|F(x^0)\|} + 1. \end{aligned}$$

□

4 Numerical Results

In this section, we present two numerical examples to verify the effectiveness of the proposed algorithms. Also, we compare our algorithms with the some existing ones. Numerical experiments were conducted using Matlab version R2016, running on a PC with CPU i3 and 10GB Ram.

Example 4.1. We compare Algorithm 3.1 with the algorithm (1.2) (shortly, `T.N.Hai`) given by Trinh Ngoc Hai and the algorithm (1.1) (shortly, `B.C`) given by Bello Cruz and Isuem. Let $H = \mathbb{R}^n, F(x) = \Big(\sin(\|x\|) + 2\Big)x$, for all $x \in \mathbb{R}^n$. The feasible set is $C = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.
We can see all the conditions of the algorithms are satisfied. In all the algorithms, we use the same stoping rule $\|x^k - x^*\| \leq 10^{-4}$, where $x^* = 0$ is the unique solution of the problem, the same starting point x_0, which is randomly generated. We compare the algorithms with the different λ_k. The results are presented in Table 1.

Table 1: Comparison of Algorithm 3.1 with `T.N.Hai` and `B.C`, (-) means λ_k is not satisfy.

	T.N.Hai		`Algorithm 3.1`		`B.C`	
	`Times(s)`	`Iter.`	`Times(s)`	`Iter.`	`Times(s)`	`Iter.`
$\frac{1}{k^{0.1}}$	0.0084	19	0.0060	52	(-)	(-)
$\frac{1}{k^{0.2}}$	0.0077	12	0.0065	16	(-)	(-)
$\frac{1}{k^{0.5}}$	0.0065	8	0.0044	8	(-)	(-)
$\frac{1}{k^{0.6}}$	0.0073	13	0.0043	8	0.0152	13
$\frac{1}{k^{0.7}}$	0.0091	19	0.0056	14	0.0162	11
$\frac{1}{k^{0.8}}$	0.0069	33	0.0044	17	0.0139	11
$\frac{1}{k^{0.9}}$	0.0083	74	0.0044	25	0.0135	33
$\frac{1}{ln(100k+1)}$	0.0066	35	0.0050	32	(-)	(-)

As we can see from this table, the computational time of Algorithm 3.1 are much smaller than those of `T.N.Hai` and `B.C`.

Example 4.2. Let H be an Hilbert space, $C = \{x \in H : \|x\| \leq 1\}$, mapping $F : C \to C$ is defined by

$$F(x) = \begin{cases} \left(\frac{1}{\|x\|} - \frac{1}{2}\right)x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We will show that F is strongly psedoumonotone on C.
For all $x, y \in C$ satisfying $\langle F(x), y - x\rangle \geq 0$, we obtain $\langle x, y - x\rangle \geq 0$. We have

$$\begin{aligned} \langle F(y), y - x\rangle &= \left(\frac{1}{\|y\|} - \frac{1}{2}\right)\langle y, y - x\rangle \\ &\geq \left(\frac{1}{\|y\|} - \frac{1}{2}\right)(\langle y, y - x\rangle - \langle x, y - x\rangle) \\ &\geq \frac{1}{2}\|y - x\|^2. \end{aligned}$$

Next, we apply Alogrithm 3.2 to problem VIP(F,C), using the stopping rule $\|x^k - x^*\| \leq 10^{-2}$, where $x^* = 0$ is the unique solution of problem VIP(F,C). We have

$$\begin{cases} \|F(x^0)\| = \dfrac{2-\|x^0\|}{2} = 0.0274, \\ M = \sup\left\{\dfrac{2-\|x\|}{2} : x \in C\right\} = 1, \\ \lambda := \dfrac{1}{4}\left(\sqrt{\dfrac{2}{\gamma} + 4\dfrac{\epsilon}{M}} - \sqrt{\dfrac{2}{\gamma}}\right)^2 = 2.488 \times 10^{-5}. \end{cases}$$

Using the formula provided in Theorem 3.2, we caculate the maximum number of steps is 273489. In fact, Alogirthm 3.2 stops after 273236 steps.

5 Conclusion

We have presented in this paper the gradient projection algorithm for solving strongly pseudomonotone variational inequalities. We establish convergence of these algorithms without Lipschitz continuity assumption. The strong convergence of the methods is proved and the numerical illustration is given.

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