TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

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STABILITY OF VOLTERA EQUATION ON TIME SCALES

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https://doi.org/10.51453/2354-1431/2021/627

Article info

Recieved:
25/10/2021
Accepted:
20/11/2021

Keywords:
Robust stability, Volterra differential equations, Bohl-Perron theorem.

Abstract:

In this paper, we develop a robust stability theorem for Volterra equations on time scales. We prove that these equations are preserved the boundedness and exponential stability under perturbations. The findings can be considered as a generalization for the robust stability of differential and difference Volterra equations.

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TÍNH ỔN ĐỊNH CỦA PHƯƠNG TRÌNH VOLTERA TRÊN THANG THỜI GIAN

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https://doi.org/10.51453/2354-1431/2021/627

Thông tin bài báo

Ngày nhận bài:
25/10/2021
Ngày duyệt đăng:
20/11/2021

Từ khóa:
Tính ổn định vững, Phương trình vi phân Volterra, Định lý Bohl-Perron

Abstract:

Trong bài báo này, chúng ta phát triển định lý về tính ổn định vững cho lớp các phương trình Volterra trên thang thời gian. Chúng ta chứng minh được rằng, dưới tác động của nhiễu, nghiệm của những phương trình này bảo toàn tính bị chặn và tính ổn định vững. Kết quả đạt được cũng được xem như là sự khái quát hóa của tính ổn định vững của phương trình vi phân và phương trình sai phân Volterra.

1 Introduction

Studying the robust stability of systems plays an important role both in theory and practice. Since the system always operates under the effect of uncertain perturbations. The designers want to have systems working stably under small perturbations. If the system is described by mathematical models, the study of its robust stability via analyzing parameters is an interesting problem. There are many works dealing with conditions imposed on coefficients under which the system is robustly stable. For example, one can measure the robust stability by using the so-called stability radii for linear systems [6]. However, it is difficult to compute the stability radius of a time-varying system, which leads to consider conditions of perturbations under which some the stability of perturbed systems is preserved (see [7, 12]).

To unify the presentation in difference equations and differential equations of the results for studied above problems, Tien and Du in [8] have considered ordinary dynamic equations on time scale $\mathbb{T}$. They have proved that under small perturbations, the perturbed dynamic equations preserve the exponential stability if the original systems are exponentially stable. The aim of this paper is to continue the study of this problem by considering the robust stability for the Volterra systems under the form

$$x^{\Delta}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)\Delta s + f(t), \tag{1.1}$$

for all $t \geq 0, t \in \mathbb{T}$, where $A(\cdot)$ and $H(\cdot,\cdot)$ are specified later. We deal with the preservation of the stability for this dynamic equation under small perturbations. Since the derivative of state process $x(t)$ at time t depends on all past path $x(s), 0 \leq s \leq t$, we have to use a more general inequality of Gronwall-Bellman type to obtain the upper bound of perturbations.

The paper is organized as follows. In the next section we recall some basic notions and preliminary results on time scales. In section 2, we present the properties of linear Volterra equation. Finally, in section 3, we prove that if the linear Volterra equations are exponentially stable, then under small Lipschitz perturbations they are still exponentially stable.

2 Linear Voltera differential equations on time scales

2.1 Time scales

In recent years, to unify continuous and discrete analysis or to describe the processing of numerical calculation with non-constant steps, a new theory was born and is more and more extensively concerned, that is the theory of the analysis on time scales, which was introduced by Stefan Hilger 1988 (see [1]). A time scale is a nonempty closed subset of the real numbers, enclosed with the topology inherited from the standard topology on $\mathbb{R}$. We usually denote it by the symbol $\mathbb{T}$. On the time scale $\mathbb{T}$, we define the forward jump operator $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the graininess $\mu(t) = \sigma(t) - t$. Similary, the backward operator is defined as $\varrho(t) = \sup\{s \in \mathbb{T} : s < t\}$ and the backward graininess is $\mu(t) = t - \varrho(t)$. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\varrho(t) = t$, left-scattered if $\varrho(t) < t$ and isolated if t is simultaneously right-scattered and left-scattered.

A regulated function f is called *rd-continuous* if it is there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point and continuous at every right-dense point. The set of rd-continuous functions defined on the interval J valued in X will be denoted by $\mathrm{C_{rd}}(J, X)$. A function f from $\mathbb{T}$ to $\mathbb{R}$ is *regressive* (resp., *positively regressive*) if for every $t \in \mathbb{T}$, then $1+\mu(t)f(t) \neq 0$ (resp., $1+\mu(t)f(t) > 0$). We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T},\mathbb{R})$ (resp., $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T},\mathbb{R})$) the set of (resp., positively regressive) regressive functions, and $\mathrm{C_{rd}}\mathcal{R}(\mathbb{T},\mathbb{R})$ (resp., $\mathrm{C_{rd}}\mathcal{R}^+(\mathbb{T},\mathbb{R})$) the set of rd-continuous (resp., positively regressive) regressive functions from $\mathbb{T}$ to $\mathbb{R}$. For all $x, y \in \mathbb{T}$, we define the *circle plus* and the *circle minus*:

$$p \oplus q := p + q + \mu(t)pq, \quad p \ominus q := \frac{p-q}{1+\mu(t)q}.$$

It is easy to verify that, for all $p, q \in \mathcal{R}$, $p \oplus q, p \ominus q, \ominus p, \ominus q \in \mathcal{R}$. Element $(\ominus q)(\cdot)$ is called the inverse element of element $q(\cdot) \in \mathcal{R}$. Hence, the set $\mathcal{R}(\mathbb{T},\mathbb{R})$ with the calculation $\oplus$ forms an Abelian group.

Definition 2.1 (Delta derivative)**.** *A function $\varphi : \mathbb{T} \to R^d$ is called delta differentiable* *at t if there exists a vector $\varphi^{\Delta}(t)$ such that for all $\varepsilon > 0$,*

$$\|\varphi(\sigma(t)) - \varphi(s) - \varphi^{\Delta}(t)(\sigma(t) - s)\| \le \varepsilon|\sigma(t) - s|$$

for all $s \in (t-\delta, t+\delta) \cap T$ and for some $\delta > 0$. The vector $\varphi^{\Delta}(t)$ is called the delta derivative of f at t.

2.2 Exponential Functions

Let $\mathbb{T}$ be an unbounded above time scale, that is $\sup \mathbb{T} = \infty$.

Definition 2.2 (Exponential stability). *Let $p : \mathbb{T} \to \mathbb{R}$ is regressive, we define the exponential function by*

$$e_p(t, t_0) = \exp\Big\{ \int_{t_0}^{t} \lim_{h \searrow \mu(s)} \frac{\mathrm{Ln}(1 + hp(s))}{h} \Delta s \Big\},$$

where $\mathrm{Ln}\, a$ is the principal logarithm of the number a.

We state properties of the exponential function as follow: If p, q are regressive, rd-continuous functions and $t, r, s \in \mathbb{T}$ then the following hold:

$$e_0(t, s) = 1, \text{and } e_p(t, t) = 1;$$
$$e_p(t, s)e_q(t, s) = e_{p+q}(t, s).$$
$$e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s);$$
$$e_p(t, s)e_p(s, r) = e_p(t, r).$$
$$\frac{1}{e_p(t, s)} = e_{-p}(t, s) = e_p(s, t);$$
$$\frac{e_p(t, s)}{e_q(t, s)} = e_{p-q}(t, s).$$

Theorem 2.3 (see [1]). *If p is regressive and $t_0 \in \mathbb{T}$, then $e_p(., t_0)$ is a unique solution of the initial value problem*

$$x^{\Delta}(t) = p(t)x(t),\ x(t_0) = 1.$$

Let $\mathbb{T}$ be time scale that is unbounded above. For any $a, b \in \mathbb{R}$, the notation $[a, b]$ or (a, b) means the segment on $\mathbb{T}$, that is $[a, b] = \{t \in \mathbb{T} : a \le t \le b\}$ or $(a, b) = \{t \in \mathbb{T} : a < t < b\}$ and $\mathbb{T}_a = \{t \ge a : t \in \mathbb{T}\}$. We can define a measure $\Delta_{\mathbb{T}}$ on $\mathbb{T}$ by considering the Caratheodory construction of measures when we put $\Delta_{\mathbb{T}}[a, b) = b - a$. The Lebesgue integral of a measurable function f with respect to $\Delta_{\mathbb{T}}$ is denoted by $\int_a^b f(s)\Delta_{\mathbb{T}} s$ (see [2]).

The Gronwall-Bellman's inequality will be introduced and applied in this paper.

Lemma 2.4 (see [9]). *Let the functions $u(t), \varphi(t), v(t),\ w(t, r)$ be nonnegative and continuous for $a \le \tau \le r \le t$, and let c_1 and c_2 be nonnegative. If for all $t \in \mathbb{T}_a$*

$$u(t) \le \varphi(t)\Big\{c_1 + c_2 \int_{\tau}^{t} \Big[v(s)u(s) + \int_{\tau}^{s} w(s, r)u(r)\Delta r\Big]\Delta s\Big\},$$

then for all $t \ge \tau$,

$$u(t) \le c_1\varphi(t)e_{p(\cdot)}(t, \tau),$$

where $p(\cdot) = c_2\left[v(\cdot)\varphi(\cdot) + \int_{\tau}^{\cdot} w(\cdot, r)\varphi(r)\Delta r\right]$.

In the whole paper, the time variable t will be omited for brevity, if it does not cause misunderstanding. For any function $g(t)$ defined on the time scale $\mathbb{T}$, we write $g_{\sigma}(t)$ for $g(\sigma(t))$.

2.3 Solution of linear Volterra differential equations

Let $a \in \mathbb{T}$ be a fixed point. Let X be a Banach space and $\mathcal{L}(X)$ be the space of the continuous linear transformations on X. We consider the linear Volterra equation

$$x^{\Delta}(t) = A(t)x(t) + \int_0^t H(t, s)x(s)\Delta s + q(t), \tag{2.1}$$

for all $t \ge a$, with the initial condition $x(t_0) = x_0 \in X$, where $A(\cdot) : \mathbb{T}_a \to \mathcal{L}(X)$ is a continuous function; $H(\cdot, \cdot)$ is a two variable continuous function defined on the set

$\{(t,s): t,s \in \mathbb{T}_a \text{ and } t_0 \le s \le t < \infty\}$, valued in $\mathcal{L}(X)$ and $q: \mathbb{T}_a \to X$ is a continuous function. The existence and uniqueness of solutions to (2.1) can be proved by a similar manner as in [4].

The homogeneous equation corresponding with (2.1) is

$$x^{\Delta}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)\Delta s, \quad (2.2)$$

for all $t \ge a$. We define the Cauchy operator $\Phi(t,s), t \ge s \ge t_0$ generated by the system (2.2) as the solution of the equation

$$\begin{cases} \Phi^{\Delta}(t,s) = A(t)\Phi(t,s) + \int_s^t H(t,\tau)\Phi(\tau,s)\Delta\tau, \\ \Phi(s,s) = I, \ t \ge s \ge a. \end{cases} \quad (2.3)$$

In the following, we stipulate that $\Phi(t,s) = 0$ if $t < s$. With this convention we have the following useful lemma, called the variation of constants formula,

Lemma 2.5. *The solution of the Volterra equation* (2.1) *can be expressed as*

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\sigma(s))q(s)\Delta s, \quad (2.4)$$

for all $t > t_0$.

Chứng minh. By directly differentiating both sides of (2.4) we get

$$\begin{aligned} x^{\Delta}(t) &= \Big(A(t)\Phi(t,t_0) \\ &+ \int_{t_0}^t H(t,\tau)\Phi(\tau,t_0)\Delta\tau\Big)x_0 + q(t) \\ &+ \int_{t_0}^t \Big[A(t)\Phi(t,\sigma(s)) \\ &+ \int_s^t H(t,\tau)\Phi(\tau,\sigma(s))\Delta\tau\Big]q(s)\Delta s \\ &= A(t)\Big(\Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\sigma(s))q(s)\Delta s\Big) \\ &+ \int_{t_0}^t H(t,\tau)\Phi(\tau,t_0)x_0\Delta\tau \\ &+ \int_{t_0}^t q(s)\Delta s \int_s^t H(t,\tau)\Phi(\tau,\sigma(s))\Delta\tau + q(t) \\ &= A(t)x(t) + \int_{t_0}^t H(t,\tau)\Big(\Phi(\tau,t_0)x_0 \\ &+ \int_{t_0}^{\tau} \Phi(\tau,\sigma(s))q(s)\Delta s\Big)\Delta\tau + q(t) \\ &= A(t)x(t) + \int_{t_0}^t H(t,\tau)x(\tau)\Delta\tau + q(t). \end{aligned}$$

The proof is complete. □

We note that for the Volterra equation (2.2) the semi-group property of the Cauchy operator in general is not true. Indeed, by definition

$$\begin{aligned} \Phi^{\Delta}(t,s) &= A(t)\Phi(t,s) + \int_s^t H(t,\tau)\Phi(\tau,s)\Delta\tau \\ &= A(t)\Phi(t,s) + \int_u^t H(t,\tau)\Phi(\tau,s)\Delta\tau + q_s(t), \end{aligned}$$

where

$$q_s(t) = \int_s^u H(t,\tau)\Phi(\tau,s)\Delta\tau.$$

Therefore, by applying (2.4) it follows that

$$\begin{aligned} \Phi(t,s) &= \Phi(t,u)\Phi(u,s) + \int_u^t \Phi(t,\sigma(h))q_s(h)\Delta h \\ &= \Phi(t,u)\Phi(u,s) \\ &+ \int_u^t \Phi(t,\sigma(h)) \int_s^u H(h,\tau)\Phi(\tau,s)\Delta\tau\Delta h. \end{aligned}$$

Thus, the semi-group property

$$\Phi(t,s) = \Phi(t,u)\Phi(u,s)$$

is true if and only if

$$\int_u^t \Phi(t,\sigma(h)) \int_s^u H(h,\tau)\Phi(\tau,s)\Delta\tau\Delta h = 0$$

for all $a \le s \le u \le t$, which implies that $H(t,s) = 0$ for all $t \ge s \ge a$.

This fact tells us that the classical method using semi-group property to study the Bohl-Perron Theorem for Volterra equations is no longer valid because this method profits the semi-group property to obtain an inequality by which we can prove the exponential stability of the unperturbed equation (see [7, 8] for examples).

Definition 2.6. *Let $t \ge t_0$, $t, t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$ and $\omega > 0$,*

i) *The Volterra equation* (2.2) *is uniformly bounded if there exists a positive number M_0, independent of t_0, such that*

$$\|x(t,t_0,x_0)\| \le M_0 \|P(t_0)y_0\|, \qquad (2.5)$$

ii) *The Volterra equation* (2.2) *is said to be ω-exponentially stable if there exists a positive number M, independent of t_0 such that*

$$\|x(t,t_0,x_0)\| \le M \|P(t_0)y_0\| e_{\ominus\omega}(t,t_0), \qquad (2.6)$$

The following characterizations of uniform stability and exponential stability are straightforward generalizations of the well-known results for ordinary differential equations, see the proof of (3.5) and (4.13) in [3]. Therefore, we omit the details of the proof.

Theorem 2.7. *There hold the following statements*

i) *The Volterra equation* (2.2) *is uniformly bounded if and only if there exists a positive number M_0 such that*

$$\|\Phi(t,s)\| \le M_0, \; t \ge s \ge a. \qquad (2.7)$$

ii) *Let ω is positive. The Volterra equation* (2.2) *is ω-exponentially stable if and only if there exists a positive number M such that*

$$\|\Phi(t,s)\| \le M e_{\ominus\omega}(t,s), \; t \ge s \ge a. \qquad (2.8)$$

3 Stability of Volterra equation under small perturbations

In this section, we consider the effect of small perturbations to the stability of the Volterra equation (2.2). Let $H(\cdot,\cdot)$ be a continuous kernel defined onthe set $\{(t,s) : t \ge s, t, s \in \mathbb{T}_0\}$. Suppose that for every $t \ge s$ and $x \in X$, the coefficients $H(t,s)x$ and $A(t)x$ of the equation (2.2) are perturbed by noise and they become $H(t,s)x \hookrightarrow H(t,s)x + f(t,s,x)$ and $A(t)x \hookrightarrow A(t)x + g(t,x)$. Thus, for any $t_0 \in \mathbb{T}_0$, the Cauchy problem for the perturbed equation (2.2) has the form

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} H(t,s)x(s)\Delta s \\ \qquad + \int_{t_0}^{t} f(t,s,x(s))\Delta s + g(t,x(t)), \\ x(t_0) = x_0 \in X, t \ge t_0 \end{cases} \qquad (3.1)$$

where $f(t,s,x)$ and $g(t,x)$ are continuous functions, Lipschitz in x with Lipschitz coefficients $k_{t,s}$ and l_t respectively, where $k_{t,s}, t \ge s \ge 0$ and $l_t, t \ge 0$ are continuous functions.

For any $x_0 \in X$ and $t_0 \ge 0$, the equation (3.1) has a unique solution, namely $x(\cdot, t_0, x_0)$, with the initial condition $x(t_0, t_0, x_0) = x_0$ and this solution is defined on $t \ge t_0$. The proof of this fact can be done by using Picard approximations (see [4, Theorem 3]). Suppose further that

$$f(t,s,0) = 0; \quad g(t,0) = 0, \text{ for all } t \ge s \ge 0.$$

With these assumptions, the equation (3.1) has the trivial solution $x(\cdot) \equiv 0$.

In the following, we write simply $x(\cdot)$ or $x(\cdot, t_0)$ for $x(\cdot, t_0, x_0)$ if there is no confusion. The robust stability for the system (3.1) when $\mathbb{T} = \mathbb{R}$ under small perturbations has been studied by T.A. Burton in [5] and R. Grimmer et al. in [10] via Lyapunov functions. S. I.

Grossman et al., in [11], considered the uniform stability of (3.1) with the functions f and g to be “hight order" by direct estimates.

In this paper, we develop the robust stability dealing with in $\mathbb{R}$ to the arbitrary times scale. Further, we approach the problem by an other technique. We will use the general Gronwall-Bellman inequality to give conditions under which the solution of the system (3.1) is either bounded or exponentially stable. To proceed, we need the following Lemma 2.4

Firstly, we consider the boundedness of solutions of the equation (2.2) under small perturbations. For convenience, we denote $\gamma_{t,s} = \int_s^t k_{t,u}\Delta u,\ t \geq s \geq a$.

Theorem 3.1. *Assume that the equation* (2.2) *is uniformly bounded and*

$$N = \int_{t_0}^{\infty} (l_t + \gamma_{t,t_0})\,\Delta t < \infty.$$

Then, there exists a constant $M_1 > 0$ *such that the solution* $x(\cdot)$ *of* (3.1) *satisfies*

$$\|x(t)\| \leq M_1 \|x(t_0)\|, \quad t \geq t_0. \tag{3.2}$$

Chứng minh. From the variation of constants formula (2.4), it follows that

$$\begin{aligned} x(t) &= \Phi(t,t_0)x(t_0) \\ &\quad + \int_{t_0}^{t} \Phi(t,\sigma(\tau))\Big(g(\tau,x(\tau)) \\ &\quad + \int_{t_0}^{\tau} f(\tau,u,x(u))\Delta u\Big)\Delta\tau, \end{aligned} \tag{3.3}$$

for all $t \geq t_0$. By virtue of Lipschitz condition of $f(t,s,x)$, $g(t,x)$ in x and the boundedness assumption of solutions (see Definition 2.7), we get

$$\begin{aligned} &\|x(t)\| \leq M_0 \|x(t_0)\| \\ &+ M_0 \int_{t_0}^{t} \Big(l_\tau \|x(\tau)\| + \int_{t_0}^{\tau} k_{\tau,u} \|x(u)\| \Delta u \Big) \Delta\tau, \end{aligned}$$

for all $t \geq t_0$. By using generalized Gronwall-Bellman inequality in Lemma 2.4 with $\varphi = 1,\ c_1 = M_0 \|x(t_0)\|$ and $c_2 = M_0$ we have

$$\|x(t)\| \leq M_0 \|x(t_0)\| e_{p(\cdot)}(t,t_0),$$

where

$$p(\tau) = M_0\big(l_\tau + \int_{t_0}^{\tau} k_{\tau,u}\Delta u\big).$$

Since $p(\tau)$ is positive,

$$\begin{aligned} e_{p(\cdot)}(t,t_0) &\leq \exp\Big(\int_t^{t_0} p(\tau)\Delta\tau\Big) \\ &= \exp\Big(M_0 \int_t^{t_0} \big(l_\tau + \int_{t_0}^{\tau} k_{\tau,u}\Delta u\big)\Delta\tau\Big) \\ &\leq e^{M_0 N}. \end{aligned}$$

Therefore, we get (3.2)

$$\|x(t)\| \leq M_0 e^{M_0 N} \|x(t_0)\|,\ t \geq t_0.$$

The proof is complete. □

Next, we consider the robust exponential stability of (2.2). We will show that the Volterra equation (2.2) preserves the exponential stability under small perturbations.

With ω is a regressive number, denote

$$\varphi_{t,s} = \int_s^t e_\omega(t,u) k_{t,u}\Delta u,\ t \geq s \geq 0.$$

Then, we have the following theorem

Theorem 3.2. *Assume that the equation* (2.2) *is* ω*-exponentially stable and*

$$\limsup_{t\to\infty} (l_t + \varphi_{t,t_0}) = \delta < \frac{\omega}{M(1+\mu^*\omega)}. \tag{3.4}$$

Then, there exist positive constants $K,\ \omega_1$ *such that*

$$\|x(t)\| \leq K e_{\ominus\omega_1}(t,s)\|x(s)\|,$$

for all $t \geq s \geq t_0$*, where* $x(\cdot) = x(\cdot,s)$ *is the solution of* (3.1)*, with the initial condition* $x(s)$*. That is, the perturbed equation* (3.1) *is* ω_1*-exponentially stable.*

Chứng minh. Let ε_0 be a positive number such that

$$\delta + \varepsilon_0 \leq \frac{\omega}{M(1+\mu^*\omega)}$$

Then, from (3.4), there exists a number $T_0 > 0$ in $\mathbb{T}$ such that

$$l_t + \varphi_{t,t_0} < \delta + \varepsilon_0, \;\; t \geq T_0. \qquad (3.5)$$

By the continuity of solutions of (3.1) on the initial condition we can find a positive constant M_{T_0}, depending only on T_0 such that

$$\|x(t)\| \leq M_{T_0}\|x(s)\|, \;\; t_0 \leq s \leq t \leq T_0. \qquad (3.6)$$

By formula (3.3), estimate (2.8) we get

$$\begin{aligned}\|x(t)\| &\leq \|\Phi(t,s)x(s)\| \\ &+ \int_s^t \big\|\Phi\big(t,\sigma(\tau)\big)\big\|\Big(\|g(\tau,x(\tau))\| \\ &+ \int_s^\tau \|f(\tau,u,x(u))\|\Delta u\Big)\Delta\tau \\ &\leq Me_{\ominus\omega}(t,s)\|x(s)\| \\ &+ M\int_s^t e_{\ominus\omega}\big(t,\sigma(\tau)\big)\Big(l_\tau\,\|x(\tau)\| \\ &+ \int_s^\tau k_{\tau,u}\,\|x(u)\|\,\Delta u\Big)\Delta\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x(t)\|e_\omega(t,s) &\leq M\|x(s)\| \\ &+ M\int_s^t e_\omega(\sigma(\tau),s)\Big(l_\tau\,\|x(\tau)\| \\ &+ \int_s^\tau k_{\tau,u}\,\|x(u)\|\,\Delta u\Big)\Delta\tau \\ &\leq M\|x(s)\| \\ &+ M\int_s^t (1+\omega\mu(\tau))\Big(l_\tau\|x(\tau)\|e_\omega(\tau,s) \\ &+ \int_s^\tau e_\omega(\tau,u))k_{\tau,u}\|x(u)\|e_\omega(u,s)\Delta u\Big)\Delta\tau.\end{aligned}$$

Using the generalized Gronwall-Bellman inequality in Lemma 2.4 with $\varphi(t) = M$, $c_1 = \|x(s)\|$, $c_2 = 1$, and $v(t) = e_\omega(\sigma(\tau),\tau)l_t$, it follows that

$$e_\omega(t,s)\|x(t)\| \leq M\|x(s)\|\,\mathrm{e}_{q(\cdot)}(t,s), \qquad (3.7)$$

where

$$q(\tau) = M[1+\mu(\tau)\omega]\Big(l_\tau + \int_s^\tau e_\omega(\tau,u)k_{\tau,u}\Delta u\Big). \qquad (3.8)$$

First, we consider the case $t_0 \leq s \leq T_0 < t$. Since $q(t)$ is positive, by the definition of the exponential function it follows that

$$\begin{aligned}\|x(t)\| &\leq M\|x(s)\|e_{\ominus\omega}(t,s)\,\mathrm{e}_{q(\cdot)}(t,s) \\ &= M\|x(s)\|e_{\ominus\omega}(t,s)\,\mathrm{e}_{q(\cdot)}(t,T_0).\,\mathrm{e}_{q(\cdot)}(T_0,s) \\ &\leq M\|x(s)\|e_{\ominus\omega}(t,s)\,\mathrm{e}_{q(\cdot)}(t,s).\,\mathrm{e}_{q(\cdot)}(T_0,t_0) \\ &\leq M\|x(s)\|\,\mathrm{e}_{q(\cdot)}(T_0,t_0)\,\mathrm{e}_{q\ominus\omega}(t,s).\end{aligned}$$

Combining with (3.5) and (3.8) we have

$$\begin{aligned}q\ominus\omega &= M[1+\mu(\tau)\omega] \\ &\times \Big(l_\tau + \int_s^\tau e_\omega(\tau,u)k_{\tau,u}\Delta u\Big)\ominus\omega \\ &\leq M[1+\mu(\tau)\omega](l_\tau+\varphi_{\tau,t_0})\ominus\omega \\ &\leq M(\delta+\varepsilon_0)[1+\mu(\tau)\omega]\ominus\omega \\ &= \frac{M(\delta+\varepsilon_0)[1+\mu(\tau)\omega]-\omega}{1+\mu(\tau)\omega} \\ &= -\frac{\omega_1}{1+\mu(\tau)\omega} = \ominus\omega_1\end{aligned}$$

where $\omega_1 := \omega - M(\delta+\varepsilon_0)[1+\mu^*\omega] > 0$. Thus,

$$\|x(t)\| \leq K_1 e_{-\omega_1}(t,s)\|x(s)\|,$$

where $K_1 = M\,\mathrm{e}_{q(\cdot)}(T_0,t_0)$.

Next, in case $t_0 < T_0 \leq s \leq t$, using a similar argument as above we get

$$\|x(t)\| \leq M\|x(s)\|e_{-\omega_1}(t,s).$$

Consider the remaining case $t_0 \leq s \leq t \leq T_0$. With $\omega_1 > 0$ defined above and from the inequality (3.6), we have

$$\begin{aligned}\|x(t)\| &\leq M_{T_0}\|x(s)\| \\ &\leq M_{T_0}e_{\omega_1}(t,s)e_{\ominus\omega_1}(t,s)\|x(s)\| \\ &\leq M_{T_0}e_{\omega_1}(T_0,t_0)e_{\ominus\omega_1}(t,s)\|x(s)\|.\end{aligned}$$

Combining the above estimates yields

$$\|x(t)\| \leq Ke_{\ominus\omega_1}(t,s)\|x(s)\| \text{ for all } t \geq s \geq t_0,$$

where $K = \max\{M, K_1, M_{T_0}e_{\omega_1}(T_0, t_0)\}$. The proof is complete. □

For the Volterra equations with bounded memory we have the following assessment.

Corollary 3.3. *Suppose that the equation* (2.2) *is ω-exponentially stable and there exists a positive constant β such that $k_{t,s} = 0$ when $t - s > \beta$. Then, the inequality*

$$\limsup_{t\to\infty}\left(l_t + e^{\omega\beta}\int_{0\vee(t-\beta)}^{t} k_{t,u}\Delta u\right) = \delta < \frac{\omega}{M}$$

implies the exponential stability of the equation (3.1).

Chứng minh. Since $k_{t,s} = 0$ when $t - s > \beta$, we have

$$\varphi_{t,s} = \int_s^t e_\omega(t,u)k_{t,u}\Delta u \leq e^{\omega\beta}\int_{0\vee(t-\beta)}^{t} k_{t,u}\Delta u.$$

Therefore,

$$\limsup_{t\to\infty}(l_t + \varphi_{t,s}) \leq \limsup_{t\to\infty}\left(l_t + e^{\omega\beta}\int_{0\vee(t-\beta)}^{t} k_{t,u}\Delta u\right) \leq \delta.$$

The proof is complete. □

Remark 3.4. *In case there is only outer force perturbation intervening into the equation* (2.2), *i.e., $k_{t.s} = 0$, the condition* (3.4) *becomes*

$$\limsup_{t\to\infty} l_t = \delta < \frac{\omega}{M}.$$

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