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BOHL THEOREM FOR VOLTERRA EQUATION ON TIME SCALES

Le Anh Tuan

HaNoi University of Industry, Hanoi, Vietnam

Email address: tuansl83@yahoo.com

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Abstract:

This paper is concerned with the Bohl-Perron theorem for Volterra in the form equations

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + f(t),$$

on time scale $\mathbb{T}$. We will show a relationship between the boundedness of the solution of Volterra equation and the stability of the corresponding homogeneous equation.

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ĐỊNH LÝ BOHL - PERRON VỀ PHƯƠNG TRÌNH VOLTERRA TRÊN THANG THỜI GIAN

Lê Anh Tuấn
Đại học Công nghiệp Hà Nội, Việt Nam
Email address: tuansl83@yahoo.com
https://doi.org/10.51453/2354-1431/2021/630

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Định lý Bohl-Perron, Phương trình vi phân Volterra, Tính bị chặn của nghiệm, Tính ổn định mũ.

Tóm tắt:

Bài báo này đề cập tới Định lý kiểu Bohl-Peron cho phương trình Volterra trên thang thời gian $\mathbb{T}$, có dạng

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + f(t).$$

Ta sẽ chỉ ra mối liên hệ giữa tính bị chặn của nghiệm của phương trình Volterra với tính ổn định của phương trình Volterra thuần nhất tương ứng.

1 Introduction

In general, it is difficult to directly study the robust stability of systems by parameters of the equations. Instead, we can estimate the output of the systems via the input and if the good input of a differential/difference equation implies the acceptable output then the system must be exponentially stable. That property is called Bohl-Perron Theorem. The earliest work in this topic belongs to Perron [1] (1930). He proved his celebrated theorem which says that if the solution of the equation $x'(t) = A(t)x(t) + f(t), t \geq 0$ with the initial condition $x(0) = 0$ is bounded for every continuous function f bounded on $[0, \infty)$, then the trivial solution of the corresponding homogeneous equation $\dot{x}(t) = A(t)x(t), t \geq 0$ is uniformly asymptotically stable. Later, one continues to study this problem for delay equation of the form $x'(t) = \sum_{k=1}^{m} A_k(t)x(t-\tau_k) + f(t)$ or $\dot{x}(t) = Lx_t + f(t), t \geq 0$ where

L is an operator acting on $C([-r,0],\mathbb{R}^n)$ (see [12] and therein). Discrete versions of Bohl-Perron Theorem can be found in [6, 7, 8].

In this paper, we extend the Bohl-Perron Theorem to a class of Volterra equations on time scales. However, the most difficulty that we face here is that the semi-group property of the Cauchy operator is no longer valid, which implies we have to find a suitable technique to solve the problem. We follow this idea by considering the exponent stability to the Volterra equations via weighted spaces $L^\gamma(\mathbb{T}_{t_0})$ and $C^\gamma(\mathbb{T}_{t_0})$ defined below. We construct an operator $\mathcal{L}$, similar to ρ in [15], and show that the exponential stability of (3.2) is equivalent the fact that $\mathcal{L}$ is surjective.

The paper is organized as follows. In the next section we recall some notion and basic properties of time scale. Section 3 present some weighted spaces and consider the solutions of Volterra equations as elements of these spaces. Finally, in section 4 we show that the exponential stability is equivalent to the surjectivity of certain operators.

2 Preliminary

A time scale is an arbitrary, nonempty, closed subset of the set of real numbers $\mathbb{R}$, denoted by $\mathbb{T}$, enclosed with the topology inherited from the standard topology on $\mathbb{R}$.

Consider a time scale $\mathbb{T}$, let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ be the *forward operator*, and then $\mu(t) = \sigma(t) - t$ be called the *graininess*; $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ be the *backward operator*, and $\nu(t) = t - \rho(t)$. We supplement $\sup\emptyset = \inf\mathbb{T}, \inf\emptyset = \sup\mathbb{T}$.

For all $x, y \in \mathbb{T}$, we define some basic calculations:

the *circle plus* $\oplus$: $x \oplus y := x + y + \mu(t)xy$;

for all $x \in \mathbb{T}$, $\ominus x := \dfrac{-x}{1+\mu(t)x}$;

the *circle minus* $\ominus$: $x \ominus y := \dfrac{x-y}{1+\mu(t)y}$.

A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered.

A function $f : \mathbb{T} \to \mathbb{R}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point.

A regulated function f is called *rd-continuous* if it is continuous at every right-dense point, and *ld-continuous* if it is continuous at every left-dense point. It is easy to see that a function is continuous if and only if it is both rd-continuous and ld-continuous. The set of rd-continuous functions defined on the interval J valued in X will be denoted by $\mathrm{C}_{\mathrm{rd}}(J,X)$.

A function $f : \mathbb{T} \to \mathbb{R}f$ from $\mathbb{T}$ to $\mathbb{R}$ is *regressive* (resp., *positively regressive*) if for every $t \in \mathbb{T}$, then $1 + \mu(t)f(t) \neq 0$ (resp., $1 + \mu(t)f(t) > 0$). We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T},\mathbb{R})$ (resp., $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T},\mathbb{R})$) the set of (resp., positively regressive) regressive functions, and $\mathrm{C}_{\mathrm{rd}}\mathcal{R}(\mathbb{T},\mathbb{R})$ (resp., $\mathrm{C}_{\mathrm{rd}}\mathcal{R}^+(\mathbb{T},\mathbb{R})$) the set of rd-continuous (resp., positively regressive) regressive functions from $\mathbb{T}$ to $\mathbb{R}$.

Definition 2.1 (Delta Derivative)**.** *A function $f : \mathbb{T} \to R^d$ is called delta differentiable at t if there exists a vector $f^\Delta(t)$ such that for all $\varepsilon > 0$,*

$$\|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in (t-\delta, t+\delta) \cap T$ and for some $\delta > 0$. The vector $f^\Delta(t)$ is called the delta derivative of f at t.

Theorem 2.2 (see [3])**.** *If p is regressive and $t_0 \in \mathbb{T}$, then the only solution of the initial value problem*

$$y^\Delta(t) = p(t),\ y(t_0) = 1$$

on $\mathbb{T}$ is defined by $e_p(t,t_0)$, say an exponential function on the time scales $\mathbb{T}$.

Let $\mathbb{T}$ be a time scale. For any $a, b \in \mathbb{R}$, the notation $[a, b]$ or (a, b) means the segment on $\mathbb{T}$, that is $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$ or $(a, b) = \{t \in \mathbb{T} : a < t < b\}$ and $\mathbb{T}_a = \{t \geq a : t \in \mathbb{T}\}$. We can define a measure $\Delta_{\mathbb{T}}$ on $\mathbb{T}$ by considering the Caratheodory construction of measures when we put $\Delta_{\mathbb{T}}[a, b) = b - a$. The Lebesgue integral of a measurable function f with respect to $\Delta_{\mathbb{T}}$ is denoted by $\int_a^b f(s)\Delta_{\mathbb{T}} s$ (see [4]).

The Gronwall-Bellman's inequality will be introduced and applied in this paper.

Lemma 2.3 (see [13]). *Let the functions $u(t), \gamma(t), v(t)$, $w(t, r)$ be nonnegative and continuous for $a \leq \tau \leq r \leq t$, and let c_1 and c_2 be nonnegative. If for $t \in \mathbb{T}_a$*

$$u(t) \leq \gamma(t) \left\{ c_1 + c_2 \int_\tau^t \left[v(s)u(s) + \int_\tau^s w(s, r)u(r)dr \right] \Delta s \right\},$$

then for $t \geq \tau$,

$$u(t) \leq c_1\gamma(t)e_{p(\cdot)}(t, \tau),$$

where $p(\cdot) = c_2 \left[v(\cdot)\gamma(\cdot) + \int_\tau^\cdot w(\cdot, r)\gamma(r)\Delta r \right]$.

3 The solution of linear Volterra equations

Let $\mathbb{T}$ be a time scale unbounded above. Suppose that the graininess function $\mu(t)$ is bounded by q constant μ^*, $0 \in \mathbb{T}$. Let X be a Banach space and $\mathcal{L}(X)$ be the space of the continuous linear transformations on X. Denote $\mathbb{T}_a = \{t \geq a : t \in \mathbb{T}\}$. For any $\gamma \geq 0$ we define

$$L^\gamma(\mathbb{T}_{t_0}) = \Big\{ f : \mathbb{T}_{t_0} \to X, f \text{ is measurable and } \int_{t_0}^\infty e_\gamma(t, t_0)\|f(t)\|\Delta t < \infty \Big\},$$

$${C_{rd}}^\gamma(\mathbb{T}_{t_0}) = \Big\{ x : \mathbb{T}_{t_0} \to X \text{ is rd-continuous, } x(t_0) = 0 \text{ and } \sup_{t \geq t_0} e_\gamma(t, t_0)\|x(t)\| < \infty \Big\},$$

with the norms defined respectively as follows

$$\|f\|_{L^\gamma(\mathbb{T}_{t_0})} = \int_{t_0}^\infty e_\gamma(t, t_0)\|f(t)\|\Delta t, \quad \text{and}$$

$$\|x\|_{C^\gamma(\mathbb{T}_{t_0})} = \sup_{\mathbb{T}_{t_0}} e_\gamma(t, t_0)\|x(t)\|.$$

It is noted that when $\gamma = 0$ we have

$$L^0(\mathbb{T}_{t_0}) = \Big\{ f : \mathbb{T}_{t_0} \to X,\ f \text{ is measurable and } \int_{t_0}^\infty \|f(t)\|\, \Delta t < \infty \Big\},$$

$$C^0_{rd}(\mathbb{T}_{t_0}) = \big\{ x : \mathbb{T}_{t_0} \to X,\ x(t_0) = 0,\ x \text{ is rd-continuous and bounded} \big\}.$$

For seeking the simplification of notations, we write $L^\gamma(\mathbb{T})$ and $C^\gamma(\mathbb{T})$ for $L^\gamma(\mathbb{T}_0), C^\gamma(\mathbb{T}_0)$ if there is no confusion.

For any $f \in L^\gamma(\mathbb{T})$, consider the linear Volterra equation

$$x^\Delta(t) = A(t)x(t) + \int_{t_0}^t K(t, s)x(s)\Delta s + f(t), \tag{3.1}$$

$t \geq t_0$, where $A(\cdot) : \mathbb{T}_a \to \mathcal{L}(X)$ is a continuous function; $K(\cdot, \cdot)$ is a two variable continuous function defined on the set $\{(t, s) : t, s \in \mathbb{T}_a \text{ and } t_0 \leq s \leq t < \infty\}$, valued in $\mathcal{L}(X)$. The existence and uniqueness of solutions to (3.1) with initial condition $x(t_0) = 0$, can be proved by similar way as in [5].

The homogeneous equation corresponding with (3.1) is

$$y^\Delta(t) = A(t)y(t) + \int_{t_0}^t K(t, s)y(s)\Delta s. \tag{3.2}$$

Since f may not be continuous, the equation (3.1) perhaps does not have the classical solution whose derivative exists every where. Therefore, we come to the concept of mild solutions as the following definition.

Definition 3.1. *The function $x(t), t \geq t_0$ is said to be a (mild) solution of* (3.1) *if*

$$x(t) = \int_{t_0}^{t}\Big(A(\tau)x(\tau) + \int_{t_0}^{\tau} K(\tau, s)x(s)\Delta s + f(\tau)\Big)\Delta\tau, \quad (3.3)$$

It is easy to see that if $x(t)$ is a mild solution of (3.1) then $x(t)$ is m_Δ–a.e differentiable in t and its derivative satisfies the equation (3.1), where a.e means "almost every where".

Assume that $\Phi(t, s), t \geq s \geq t_0$ is the Cauchy operator generated by the system (3.2), then for $t \geq s \geq t_0$, we have

$$\Phi^\Delta(t, s) = A(t)\Phi(t, s) + \int_s^t K(t, \tau)\Phi(\tau, s)\Delta\tau, \quad (3.4)$$

with $\Phi(s, s) = I$. It follows that the solution $x(t)$ of (3.1) with the initial condition $x(t_0) = 0$ is given by

$$x(t) = \int_{t_0}^{t} \Phi(t, \sigma(s))f(s)\Delta s, \ t > t_0. \quad (3.5)$$

It is easy to show that in general the Volterra equation (3.2), the Cauchy operator has no property of semi-group

$$\Phi(t, s) = \Phi(t, u)\Phi(u, s), \quad (3.6)$$

for all $0 \leq s \leq u \leq t$. That causes some difficulties in the study of Bohl-Perron theorem. To overcome, we have to find a suitable technique to solve the problem.

Lemma 3.2. *The solution $y(t, s, y_0)$ of the homogeneous equation* (3.2) *with initial condition $y(s) = y_0$ is continuous in (t, s, y_0).*

Chứng minh. It is easy to show that the solution $y(t, s, y_0), t \geq s$ is continuous in t. Thus we prove that it is continuous in (s, y_0). Let $y(t, s_0, y_0); y(t, s_1, y_1)$ be two solutions of (3.2) with initial conditions $y(s_0) = y_0$ and $y(s_1) = y_1$ respectively, where $s_0 \leq s_1 \in \mathbb{T}$; $y_0, y_1 \in Y$. First, we have

$$y(t, s_0, y_0) = y_0 + \int_{s_0}^{t} A(\tau)y(\tau, s_0, y_0)\Delta\tau + \int_{s_0}^{t}\int_{s_0}^{\tau} K(\tau, u)y(u, s_0, y_0)\Delta u\Delta\tau$$

for all $t \in [0, T]$. Therefore,

$$\|y(t, s_0, y_0)\| \leq \|y_0\| + \int_{s_0}^{t}\|A(\tau)y(\tau, s_0, y_0)\|\Delta\tau + \int_{s_0}^{t}\int_{s_0}^{\tau}\|K(\tau, u)\|\,\|y(u, s_0, y_0)\|\,\Delta u$$

which implies that

$$\|y(t, s_0, y_0)\| \leq y_0 e_{p(\cdot)}(t, s_0), \quad (3.7)$$

where $p(\cdot) = \|A(\cdot)\| + \int_{s_0}^{\cdot}\|K(\cdot, u)\|\Delta u$. Put $\varphi(t, s_0, s_1) = \|y(t, s_0, y_0) - y(t, s_1, y_1)\|$. Hence,

$$\begin{aligned}\varphi(t, s_0, s_1) &\leq \|y_0 - y_1\| \\ &+ \int_{s_0}^{s_1} \|A(\tau)\|\,\|y(\tau, s_0, y_0)\|\,\Delta\tau \\ &+ \int_{s_0}^{s_1}\int_{u}^{t} \|K(\tau, u)\|\,\|y(u, s_0, x_0)\|\,\Delta\tau\Delta u \\ &+ \int_{s_1}^{t}\|A(\tau)\|\,\varphi(u, s_0, s_1)\Delta\tau \\ &+ \int_{s_1}^{t}\int_{s_1}^{\tau}\|K(\tau, u)\|\,\varphi(u, s_0, s_1)\Delta u\Delta\tau.\end{aligned}$$

Using (3.7) we see that then there exists number $c > 0$

$$\begin{aligned}&\|\int_{s_0}^{s_1} A(\tau)y(\tau, s_0, y_0)\Delta\tau \\ &+ \int_{s_0}^{s_1}\int_{u}^{t} K(\tau, u)y(u, s_0, y_0)\Delta\tau\Delta u\| \\ &\leq c\|s_0 - s_1\|.\end{aligned}$$

By using generalized Gronwall-Bellman inequality in Lemma 2.3 with $\gamma = 1,\ c_1 = c|s_0 - s_1|, v = \|A\|, w = \|K(\tau, u)\|$ and $c_2 = 1$

$$\varphi(t, s_0, s_1) \leq (\|y_0 - y_1\| + c|s_0 - s_1|)e_{p(\cdot)}(t, \tau),$$

where $p(\cdot) = \left[v(\cdot) + \int_\tau^{\cdot} w(\cdot, r)\Delta r\right]$. We have the proof. □

Definition 3.3.

i) *The Volterra equation* (3.2) *is uniformly bounded if there exists a positive number* M_0 *such that*

$$\|\Phi(t,s)\| \le M_0,\ t \ge s \ge a. \tag{3.8}$$

ii) *Let* ω *is positive. The Volterra equation* (3.2) *is* ω*-exponentially stable if there exists a positive number* M *such that*

$$\|\Phi(t,s)\| \le M e_{\ominus\omega}(t,s),\ t \ge s \ge a. \tag{3.9}$$

4 Bohl-Perron Theorem with unbounded memory

Based on the formula (3.5) we consider the operator $\mathcal{L}_{t_0}$ defined on $L^\gamma(t_0)$ associated with the equation (3.1) as follows:

$$(\mathcal{L}_{t_0} f)(t) = \int_{t_0}^{t} \Phi(t,\sigma(s)) f(s)\Delta s, \tag{4.1}$$

for $t > t_0,\ f \in L^\gamma(t_0)$. We write simply $\mathcal{L}$ for $\mathcal{L}_0$.

Theorem 4.1. *For any* $\gamma > 0$*, if* L *maps* $L^\gamma(\mathbb{T})$ *to* $C^\gamma_{rd}(\mathbb{T})$*, then there exists a positive constant* K *such that for all* $t_0 \ge 0$,

$$\|\mathcal{L}_{t_0}\| \le K. \tag{4.2}$$

Chứng minh. First, we prove (4.2) when $t_0 = 0$. For every $t > 0$, we define an operator $F_t : L^\gamma(\mathbb{T}) \to X$ by

$$\begin{aligned} F_t(f(\cdot)) =& e_\gamma(t,0)\int_0^t \Phi(t,\sigma(s)) f(s)\Delta s \\ =& e_\gamma(t,0)\mathcal{L}f(t). \end{aligned}$$

Since $\mathcal{L}$ maps $L^\gamma(\mathbb{T})$ to $C^\gamma_{rd}(\mathbb{T})$,

$$\sup_{t\ge 0}\|F_t(f)\| = \sup_{t\ge 0} e_\gamma(t,0)\,\|\mathcal{L}f(t)\| < \infty.$$

Therefore, by the Uniform Boundedness Principle

$$\sup_{t\ge 0}\|F_t\| = K < \infty.$$

It is noted that,

$$\begin{aligned} \|\mathcal{L}\| &= \sup_{f\in L^\gamma(\mathbb{T})} \frac{\|\mathcal{L}f\|_{C^\gamma_{rd}(\mathbb{T})}}{\|f\|} \\ &= \sup_{f\in L^\gamma(\mathbb{T})} \frac{\sup_{t\ge 0}\|F_t(f)\|}{\|f\|} = \sup_{t\in\mathbb{T}0}\|F_t\| = K. \end{aligned}$$

We now prove (4.2) with arbitrary $t_0 > 0$. Let $f(t)$ be an arbitrary function in $L^\gamma(t_0)$. We define the function $\overline{f}$ as follows: $\overline{f}(t) = 0$ if $t < t_0$, else $\overline{f}(t) = f(t)$. It is seen that

$$\begin{aligned} \mathcal{L}\overline{f}(t) &= \int_0^t \Phi(t,\sigma(s))\overline{f}(s)\Delta s \\ &= \int_{t_0}^t \Phi(t,\sigma(s)) f(s)\Delta s = \mathcal{L}_{t_0} f(t), t \ge t_0. \end{aligned}$$

Therefore, from (4) we get

$$\begin{aligned} &\|\mathcal{L}_{t_0} f\|_{C^\gamma_{rd}(\mathbb{T}_{t_0})} = \sup_{t\ge t_0} e_\gamma(t,t_0)\,\|\mathcal{L}_{t_0} f(t)\| \\ &= \sup_{t\ge 0} e_\gamma(t,0)\,\|\mathcal{L}\overline{f}(t)\| = \|\mathcal{L}\overline{f}\|_{C^\gamma_{rd}(\mathbb{T})} \\ &\le K\|\overline{f}\|_{L^\gamma(\mathbb{T})} = K\|f\|_{L^\gamma(\mathbb{T}_{t_0})}. \end{aligned}$$

The proof is complete. □

Theorem 4.2. *Let* $\gamma > 0$*. The operator* $\mathcal{L}$ *maps* $L^\gamma(\mathbb{T})$ *to* $C^\gamma_{rd}(\mathbb{T})$ *if and only if* (3.2) *is* γ*-exponentially stable.*

Chứng minh. The proof contains two parts.

Necessity. First, we prove that if $\mathcal{L}$ maps $L^\gamma(\mathbb{T})$ to $C^\gamma(\mathbb{T})$ then (3.2) is γ-exponentially stable.

By virtue of Theorem 4.1, $\mathcal{L}$ is a bounded operator from $L^\gamma(\mathbb{T})$ to $C^\gamma_{rd}(\mathbb{T})$ with $\|\mathcal{L}\| = K$. For all $f \in L^\gamma(\mathbb{T})$ and $0 \le s \le t$, we have

$$\begin{aligned} & e_\gamma(t,0)\left\|\int_0^t \Phi(t,\sigma(u)) f(u)\Delta u\right\| \\ &\le \|\mathcal{L}f\|_{C^\gamma_{rd}(\mathbb{T})} \le K\,\|f\|_{L^\gamma(\mathbb{T})}. \end{aligned} \tag{4.3}$$

For any $\alpha > 0$ and $v \in X$, we consider the function

$$f_\alpha(u) = \begin{cases} \frac{1}{\alpha} e_{\ominus\gamma}(u,0)v, & \text{if } u \in [s, s+\alpha] \\ 0 & \text{if } u \notin [s, s+\alpha]. \end{cases}$$

It is seen that

$$\int_0^\infty e_\gamma(u,0)\|f_\alpha(u)\|\Delta u$$
$$= \frac{1}{\alpha}\int_s^{s+\alpha} e_\gamma(u,0)e_{\ominus\gamma}(u,0)\|v\|\Delta u = \|v\|.$$

This means that $f_\alpha \in L^\gamma(\mathbb{T})$ and $\|f_\alpha\|_{L^\gamma(\mathbb{T})} = \|v\|$. Furthermore,

$$\lim_{\alpha\to 0}\int_0^t \Phi(t,\sigma(u))f_\alpha(u)\Delta u$$
$$= \frac{1}{\alpha}\lim_{\alpha\to 0}\int_s^{s+\alpha} \Phi(t,\sigma(u))e_{\ominus\gamma}(u,0)v\Delta u$$
$$= e_{\ominus\gamma}(s,0)\Phi(t,\sigma(s))v.$$

Combining with (4.3) obtains the desired estimate

$$\|\Phi(t,\sigma(s))\| \le Ke_{\ominus\gamma}(t,s) \le Ke_{\ominus\gamma}(t,\sigma(s)),$$

for $t \ge s \ge 0$. Let $\{s_n\} \in \mathbb{T}$ such that $\sigma(s_n) \to s (n \to \infty)$,

$$\|\Phi(t,\sigma(s_n))\| \le Ke_{\ominus\gamma}(t,\sigma(s_n)),\ t \ge s \ge 0.$$

Letting $n \to \infty$ and using the continuity of solution, we obtain

$$\|\Phi(t,\sigma(s))\| \le Ke_{\ominus\gamma}(t,s),\ t \ge s \ge 0.$$

Thus, (3.2) is uniformly asymptotically stable.

Sufficiency. We will show that if (3.2) is γ-exponentially stable then $\mathcal{L}$ maps $L^\gamma(\mathbb{T})$ to $C^\gamma_{rd}(\mathbb{T})$. Let $f \in L^\gamma(\mathbb{T})$, from (4.1) we see that

$$e_\gamma(t,0)\|\mathcal{L}f(t)\|$$
$$\le Me_\gamma(t,0)\int_0^t \mathrm{e}_{\ominus\gamma}(t,\sigma(s))\|f(s)\|\Delta s$$
$$= M\int_0^t (1+\gamma\mu(s))e_\gamma(s,0)\|f(s)\|\Delta s$$
$$\le M(1+\gamma\mu^*)\|f\|_{L^\gamma(\mathbb{T})} < \infty.$$

Thus, $\mathcal{L}f \in C^\gamma_{rd}(\mathbb{T})$. The proof is complete. □

Remark 4.3. *The argument dealt with in the proof of Theorem 4.2 is still valid for $\gamma = 0$. Thus, if $\mathcal{L}$ maps L_1 to C_b then the solution of (3.2) with the initial condition $x(0) = 0$ is bounded.*

Corollary 4.4. *The equation (3.2) is γ-exponentially stable if and only if the solution of*

$$y^\Delta(t) = A(t)[1+\mu(t)\gamma]y(t) + \gamma y(t) \quad (4.4)$$
$$+ \int_0^t K(t,s)e_\gamma(\sigma(t),s)y(s)\Delta s + f(t),$$

is bounded for all $f \in L^\gamma$.

Chứng minh. Denote by $\Psi(t,s)$ the Cauchy operator of the homogeneous equation corresponding to (4.4), i.e., $\Psi(s,s) = I$ and

$$\Psi^\Delta(t,s) = A(t)[1+\mu(t)\gamma]\Psi(t,s) + \gamma\Psi(t,s)$$
$$+ \int_s^t K(t,\tau)e_\gamma(\sigma(t),\tau)\Psi(\tau,s)\Delta\tau.$$

From (3.4) we get

$$\left(e_\gamma(t,0)\Phi(t,s)\right)^\Delta = e_\gamma(\sigma(t),0)\Phi^\Delta(t,s)$$
$$+ e^\Delta_\gamma(t,0)\Phi(t,s)$$
$$= A(t)[1+\mu(t)\gamma]e_\gamma(t,0)\Phi(t,s)$$
$$+ \gamma e_\gamma(t,0)\Phi(t,s)$$
$$+ \int_s^t K(t,\tau)e_\gamma(\sigma(t),\tau)e_\gamma(\tau,0)\Phi(\tau,s)\Delta\tau$$

The uniqueness of solutions says that

$$\Psi(t,s) = e_\gamma(t,0)\Phi(t,s). \quad (4.5)$$

Hence, the γ-exponential stability of (3.2) implies that the solution of (4.4) is bounded. Let $y(t)$ be the solution of (4.4) with the initial condition $y(0) = 0$. By (4.1), this solution can be expressed as

$$y(t) = \int_0^t \Psi(t,\sigma(\tau))f(\tau)\Delta\tau = e_\gamma(t,0)\mathcal{L}f(t).$$

The boundedness of $y(t)$ says that $\mathcal{L}$ maps $\mathcal{L}^\gamma$ to C^γ. Therefore, by Theorem 4.2, the equation (3.2) is exponentially stable. The proof is complete. □

5 Bohl-Perron Theorem with damped memory

We consider the equation (3.1) with the assumption

Assumption 5.1. *$A(t)$ is bounded on $\mathbb{T}$ by a constant A and $K(t,s)$ is bounded on the set $0 \le t-s \le 1$ by N_1. Further, there is a $\beta > 0$ such that*

$$\overline{H} = \sup_{s>0} \int_s^\infty e_\beta(t,s) \|K(t,s)\| \big(\sigma(t) - s\big) \Delta t < \infty.$$

It follows from this assumption that

$$H_1 = \sup_{s \ge 0} \int_s^\infty \|K(t,s)\| \, \Delta t < \infty.$$

Denote

$$C_{1,1}(\mathbb{T}; X) = \big\{ x : \mathbb{T} \to X;\ x(0) = 0; \\ x \text{ is a.e differentiable and } \dot{x}, x \in L_1(\mathbb{T}; X) \big\}.$$

We endow $C_{1,1}(\mathbb{T}; X)$ with the norm of $L_1(\mathbb{T}; X)$. Then, it becomes an (incomplete) normed vector space. Consider the operator $\mathcal{N}$ associated to (3.1) given by

$$\mathcal{N}x(t) = x^\Delta(t) - A(t)x(t) - \int_0^t K(t,s)x(s)\Delta s,\ x \in L_1(\mathbb{T}, X). \tag{5.1}$$

For any $x \in L_1$ we have

$$\begin{aligned} &\left\| \int_0^\cdot K(\cdot, s)x(s)\Delta s \right\|_{L_1} \\ &\le \int_0^\infty \int_0^t \|K(t,s)\| \, \|x(s)\| \, \Delta s \Delta t \\ &\le \int_0^\infty \|x(s)\| \int_s^\infty \|K(t,s)\| \, \Delta t \Delta s \\ &\le H_1 \|x\|_{L_1}. \end{aligned} \tag{5.2}$$

Thus, $\mathcal{N}$ maps from $C_{1,1}$ to $L_1(\mathbb{T}; X)$. By uniqueness of solution of (3.2), it is clear that $\mathcal{N}$ is an injective map.

Theorem 5.2. *Let Assumption 5.1 holds. Then, the equation (3.2) is ω-exponentially stable for an $\omega > 0$ if and only if $\mathcal{N}$ is surjective.*

Chứng minh. Suppose that the system (3.2) is ω-exponentially stable for a certain $\omega > 0$. This means that there is a positive constant M such that $\|\Phi(t,s)\| \le M e_{\ominus\omega}(t,s)$ for any $t \ge s \ge 0$. For any $f \in L_1(\mathbb{T}, X)$ we put

$$x(t) = \mathcal{L}f(t) = \int_0^t \Phi(t, \sigma(s)) f(s) \, \Delta s.$$

It is seen that $x(t)$ is a.e differentiable and $\mathcal{N}x = f$. Further,

$$\begin{aligned} &\int_0^\infty \|x(t)\| \, \Delta t \\ &= \int_{\sigma(0)}^\infty \left\| \int_0^t \Phi(t, \sigma(s)) f(s) \Delta s \right\| \Delta t \\ &\le M \int_{\sigma(0)}^\infty \left(\int_0^t e_{\ominus\omega}(t, \sigma(s)) \, \|f(s)\| \, \Delta s \right) \Delta t \\ &= M \int_{\sigma(0)}^\infty \|f(s)\| \left(\int_{\sigma(s)}^\infty e_{\ominus\omega}(t, \sigma(s)) \Delta t \right) \Delta s. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_{\sigma(s)}^\infty e_{\ominus\omega}(t, \sigma(s)) \Delta t \\ &\quad = \int_{\sigma(s)}^\infty \frac{1 + \mu(t)\omega}{-\omega} \ominus \omega e_{\ominus\omega}(t, \sigma(s)) \Delta t \\ &\quad \le \frac{1 + \mu^*\omega}{\omega} e_{\ominus\omega}(t, \sigma(s)) \Big|_\infty^{\sigma(s)} = \frac{1 + \mu^*\omega}{\omega}. \end{aligned}$$

Thus,

$$\int_0^\infty \|x(t)\| \, \Delta t \le \frac{M(1 + \mu^*\omega)}{\omega} \|f(\cdot)\|_{L_1}.$$

Therefore, $x \in L_1(\mathbb{T}, X)$, which implies $A(\cdot)x(\cdot) \in L_1(\mathbb{T}, X)$ by virtue of boundedness of $A(\cdot)$ and

$$\int_0^{\cdot} H(\cdot, s)x(s)\Delta s \in L_1(\mathbb{T}, X)$$

by (5.2). These relations say that $x^\Delta \in L_1(\mathbb{T}, X)$. Thus, $x \in C_{1,1}(\mathbb{T}; X)$. This means that $\mathcal{N}$ is surjective.

Conversely, assume that $\mathcal{N}$ is surjective, we will show that (3.2) is ω-exponentially stable, where

$$0 < \omega < \min\left\{\beta, \frac{1}{2(1+\mu^* A + \overline{H})\,\|\mathcal{L}\|}\right\},$$

and $\beta, \overline{H}$ defined in Assumption 5.1. Indeed, since $\mathcal{N}$ is injective, we can define $\mathcal{N}^{-1}$ acting $L_1(\mathbb{T}, X)$ to $C_{1,1}(\mathbb{T}, X)$. It is clear $\mathcal{N}^{-1} = \mathcal{L}$. Moreover, by a similar way as in the proof of Theorem 4.1, we imply the boundedness of $\mathcal{L}$.

Putting $x(t) = e_{\ominus\omega}(t, 0)y(t)$, since

$$\mathcal{N}x(t) = x^\Delta(t) - A(t)x(t) - \int_0^t K(t,s)x(s)\Delta s,$$

we gets

$$\begin{aligned}\mathcal{N}x(t) &= e_{\ominus\omega}(\sigma(t), 0)y^\Delta(t) + \ominus\omega e_{\ominus\omega}(t, 0)y(t)\\ &- A(t)e_{\ominus\omega}(t,0)y(t) - \int_0^t K(t,s)e_{\ominus\omega}(s,0)y(s)\Delta s\\ &= e_{\ominus\omega}(\sigma(t), 0)\left(\mathcal{N}y(t) + Gy(t)\right).\end{aligned}$$

Let

$$\begin{aligned}G = &-\omega\big[I + \mu(t)A(t)\big]y(t) -\\ &\int_0^t K(t,s)\big[e_\omega(\sigma(t), s) - 1\big]y(s)\Delta s.\end{aligned}$$

Therefore,

$$\mathcal{N}x(t) = e_{\ominus\omega}(\sigma(t), 0)\mathcal{M}y(t), \tag{5.3}$$

where $\mathcal{M} = \mathcal{N} + G$.

Further, for any $f \in L_1(\mathbb{T}, X)$ we have

$$\begin{aligned}\int_0^\infty \|G(\mathcal{L}f)(t)\|\,\Delta t &\le \omega\big(1 + \mu^* A\big)\,\|\mathcal{L}f\|_{L_1}\\ &+ \int_0^\infty\!\!\int_0^t \|X(t,s)\|\,\Delta s\Delta t.\end{aligned}$$

with $X(t,s) = K(t,s)\Big[\big(e_\omega(t,s) - 1\big) + \mu(t)\omega e_\omega(t,s)\Big](\mathcal{L}f)(s)$. Since

$$\begin{aligned}e_\omega(t,s) - 1 &= \omega\int_s^t e_\omega(\tau, s)\Delta\tau\\ &\le \omega\int_s^t e_\omega(t,s)\Delta\tau = \omega e_\omega(t,s)(t-s).\end{aligned}$$

We have

$$\begin{aligned}&\int_0^\infty \Big\|\int_0^t X(t,s)\Delta s\Big\|\Delta t\\ &= \omega\int_0^\infty\!\!\int_0^t e_\omega(t,s)\big[(t-s) + \mu(t)\big]\\ &\qquad\times \big\|K(t,s)\mathcal{L}f(s)\big\|\Delta s\Delta t\\ &= \omega\int_0^\infty\!\!\int_0^t \mathrm{e}_\omega(t,s)\big(\sigma(t) - s\big)\\ &\qquad\times \big\|K(t,s)\mathcal{L}f(s)\big\|\Delta s\Delta t\\ &= \omega\int_0^\infty \|\mathcal{L}f(s)\|\int_s^\infty e_\beta(t,s)\big(\sigma(t) - s\big)\\ &\qquad\times \|K(t,s)\|\Delta t\Delta s\\ &= \omega\overline{K}\int_0^\infty \|\mathcal{L}f(s)\|\Delta s.\end{aligned}$$

Thus, we have

$$\int_0^\infty \|G(\mathcal{L}f)(t)\|\Delta t \le \omega\big(1 + \mu^* A + \overline{K}\big)\,\|\mathcal{L}f\|_{L_1}$$

Therefore, $G\mathcal{L}f \in L_1(\mathbb{T}, X)$ and with chosen ω as above, we obtain

$$\|G\mathcal{L}f\|_{L_1} \le \frac{\|f\|}{2},$$

which implies that $\mathcal{M}\mathcal{L} = I + G\mathcal{L}$ is invertible.

Thus, $\mathcal{M}$ is a surjective, i.e., for any $f \in L_1(\mathbb{T}, X)$, the equation

$$\mathcal{M}y = f \tag{5.4}$$

has a solution in $C_{1,1}(\mathbb{T},X)$. Using the same argument as in the proof of Theorem 4.2 we can prove that $\mathcal{M}^{-1}$ is bounded. Let $\Psi(t,s)$ be the Cauchy operator of the equation $\mathcal{M}y=0$ with the initial condition $\Psi(s,s)=I$. Then, the solution $y(t)=\mathcal{M}^{-1}f(t)$ with the initial condition $y(0)=0$ of the equation (5.4) has the expression

$$y(t)=\int_0^t \Psi(t,\sigma(s))f(s)\Delta s, \qquad t>0.$$

The bounedness of $\mathcal{M}^{-1}$ says that there is a $K_1>0$ such that $\|\mathcal{M}^{-1}f\|_{\mathcal{L}_1}\le K_1\|f\|_{L_1}$ for all $f\in L_1$, or

$$\|y(\cdot)\|_{\mathcal{L}_1}=\|\mathcal{M}^{-1}f\|_{\mathcal{L}_1}$$
$$=\int_0^\infty \Big\|\int_0^t \Psi(t,\sigma(s))f(s)\Delta s\Big\|\Delta t\le K_1\|f\|_{L_1}.$$

For any $v\in X$ and $\alpha>0$, put $f_\alpha(s)=\dfrac{1_{[0,\alpha]}(s)}{\alpha}v$, we have $\|f\|_{L_1}=\|v\|$. From above inequality, we have

$$\int_0^\infty \left\|\frac{1}{\alpha}\int_0^\alpha \Psi(t,\sigma(s))v\Delta s\right\|\Delta t\le K_1\|v\|.$$

Letting $\alpha\to 0$ obtains

$$\int_0^\infty \|\Psi(t,\sigma(0))v\|\,\Delta t\le K_1\|v\|.$$

On the other hand, since $\Psi(t,s)$ be the Cauchy operator of the equation $\mathcal{M}y=0$,

$$y^\Delta(t)-\big[\omega+(1+\mu(t)\omega)A(t)\big]y(t)$$
$$-\int_0^t K(t,s)e_\omega(\sigma(t),s)y(s)\Delta s=0.$$

We have

$$\Psi(\tau,0)^\Delta(\tau)=-\big[\omega+(1+\mu(t)\omega)A(t)\big]\Psi(\tau,0)$$
$$+\int_0^\tau K(\tau,s)e_\omega(\sigma(\tau),s)\Psi(s,0)\Delta s.$$

Then, for all $t>0$

$$\|\Psi(t,0)v\|-\|v\|$$
$$\le\int_0^t\big[\omega+(1+\mu^*\omega)A\big]\|\Psi(\tau,0)v\|\,\Delta\tau$$
$$+\int_0^t\int_0^\tau e_\omega(\sigma(\tau),s)\,\|K(\tau,s)\Psi(s,0)v\|\,\Delta s\Delta\tau.$$

Since

$$\int_0^t\int_0^\tau \|e_\omega(\sigma(\tau),s)K(\tau,s)\Psi(s,0)v\|\,\Delta s\Delta\tau$$
$$\le\int_0^\infty\|\Psi(s,0)v\|\int_s^\infty e_\omega(\sigma(\tau),s)\,\|K(\tau,s)\|\,\Delta\tau\Delta s$$
$$\le\int_0^\infty\|\Psi(s,0)v\|\int_s^\infty e_\beta(\sigma(\tau),s)\,\|K(\tau,s)\|\,\Delta\tau\Delta s,$$

and from Assumption 5.1, we have

$$\int_s^\infty e_\beta(\sigma(\tau),s)\,\|K(\tau,s)\|\,\Delta\tau$$
$$\le(1+\mu^*\beta)\int_s^\infty e_\beta(\tau,s)\,\|K(\tau,s)\|\,\Delta\tau$$
$$=(1+\mu^*\beta)\Big(\int_s^{s+1}e_\beta(\tau,s)\,\|K(\tau,s)\|\,d\tau$$
$$+\int_{s+1}^\infty e_\beta(\tau,s)\,\|K(\tau,s)\|\,\Delta\tau\Big)$$
$$\le(1+\mu^*\beta)\big(N_1e^\beta+\overline{H}\big).$$

Therefore,

$$\|\Psi(t,0)v\|\le H_2\|v\|,$$

for any $v\in X$, with $H_2=1+\big[\big(\omega+(1+\mu^*\omega)A\big)+(1+\mu^*\beta)(N_1e^\beta+\overline{H})\big]K_1$, which implies $\|\Psi(t,0)\|\le H_2$, for all $t\ge 0$. Combining this inequality with (4.5), we get

$$\|\Phi(t,0)\|\le H_2e_{\ominus\omega}(t,0),\quad t\ge 0.$$

By a similar argument we see that

$$\|\Phi(t,s)\|\le H_2e_{\ominus\omega}(t,s),\quad t\ge s\ge 0.$$

The proof is complete. □

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