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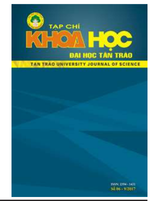

MỘT MỞ RỘNG CỦA PHƯƠNG PHÁP THÁC TRIỂN THEO THAM SỐ GIẢI HỆ PHƯƠNG TRÌNH PHI TUYẾN CÓ NHIỄU

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Phương pháp thác triển theo tham số, Hệ phương trình phi tuyến có nhiễu, Giải xấp xỉ.

Tóm tắt:

Trong bài báo này, chúng tôi đề xuất một mở rộng của phương pháp thác triển theo tham số giải hệ phương trình phi tuyến có nhiễu. Sự tồn tại và tính duy nhất nghiệm sẽ được nghiên cứu. Chúng tôi cũng thảo luận về sai số của phương pháp. Tính hiệu quả và khả năng áp dụng của phương pháp được trình bày thông qua một ví dụ.

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AN EXTENSION OF PARAMETER CONTINUATION METHOD FOR SOLVING PERTURBED SYSTEMS OF NONLINEAR EQUATIONS

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Abstract:

In this paper, we propose an extension of parameter continuation method for solving perturbed systems of nonlinear equations. The existence and uniqueness of the solution will be investigated. We also discuss error analysis of the method. The validity and applicability of the method is verified by an example.

1 Introduction

Solving systems of nonlinear equations is of great importance, because these systems frequently arise in many branches of computational mathematics. Systems of nonlinear equations are usually difficult to get their exact solutions. So many different iterative methods have been proposed to obtain approximate solutions of the systems of nonlinear equations, which were presented in [1, 6, 8, 13].

Parameter continuation method (PCM) [7, 10, 11, 14–17] is a powerful technique for solving operator equations of various kinds. In recent years, the PCM has been successfully applied to solve many nonlinear equations [3, 12, 18].

In previous paper, we studied the application of PCM for solving systems of nonlinear equations [4]. In this paper, we apply an extension of the PCM that we already presented in [5] to perturbed systems of nonlinear equations. This work can be viewed as an extension of the result in [4].

The remainder of this paper is structured as follows. In Section 2, we recall some necessary definitions and known results before introducing main results in Section 3. Section 4 draws some conclusions from the paper.

2 Preliminaries

Consider the operator equation

$$x + A(x) + B(x) = f, \tag{1}$$

where A, B are nonlinear operators from the Banach space X into itself and f is a given function in X.

Definition 2.1. (see [7]) The mapping A, which operates in the Banach space X is called monotone if for any elements $x_1, x_2 \in X$ and any $\varepsilon > 0$ the following inequality holds

$$\|x_1 - x_2 + \varepsilon\,[A(x_1) - A(x_2)]\,\| \geq \|x_1 - x_2\|. \quad (2)$$

Remark 2.2. (see [7]) If X is Hilbert space then the condition of monotony (2) is equivalent to the classical condition

$$\langle A(x_1) - A(x_2), x_1 - x_2\rangle \geq 0,\ \forall x_1, x_2 \in X,$$

where $\langle\cdot,\cdot\rangle$ is an inner product in the Hilbert space X.

Theorem 2.3. (see [5]) *Assume that A is a Lipschitz-continuous and monotone operator, B is a contractive operator. Then the equation* (1) *has a unique solution for any element $f \in X$.*

The following iteration process is constructed to find approximate solutions of the equation (1)

$$x_{i+1} = \underbrace{-\frac{1}{N}A(x_i) - \frac{1}{N}A(x_j) - \cdots - \frac{1}{N}A(x_h)}_{N\ terms} - B(x_p) + f,\ i, j, \ldots, p = 0, 1, \ldots . \quad (3)$$

The symbolic notation (3) should be understood as the following iteration processes, which consist of $N + 1$ iteration processes

$$x_{i+1} = -\varepsilon_0 A(x_i) + x_j^{(1)},\ i = 0, 1, \ldots, \quad (4a)$$

$$x_{j+1}^{(1)} = -\varepsilon_0 A G_1^{-1}(x_j^{(1)}) + x_l^{(2)},\ j = 0, 1, \ldots, \quad (4b)$$

$$\ldots, \quad (4c)$$

$$x_{p+1}^{(N)} = -B G_1^{-1} \cdots G_N^{-1}(x_p^{(N)}) + f, p = 0, 1, \ldots,$$
$$x_0^{(N)} = f, \quad (4d)$$

where $x^{(1)} = x + \varepsilon_0 A(x) \equiv G_1(x), x^{(k+1)} = x^{(k)} + \varepsilon_0 A G_1^{-1} \cdots G_k^{-1}(x) \equiv G_{k+1}(x),\ k = 1, 2, \ldots, N-2$. Assume that the number of steps in each iteration scheme of iteration processes (4a)–(4d) is the same and equals n. Let x_n be approximate solutions of the equation 1. Note that x_n depends on N. Hence we denote $x(n, N) \equiv x_n$. We have the following theorem.

Theorem 2.4. (see [5]) *Let the assumptions of Theorem 2.3 are satisfied. Then the sequence of approximate solutions $\{x(n, N)\}$, $n = 1, 2, \ldots$ constructed by iteration processes* (4a)–(4d) *converges to the exact solution $x \in X$ of the equation* (1). *Moreover, the following estimates hold*

$$\|x(n,N) - x\| \leq \frac{1}{1 - q_B}\left[\frac{q_A^{n+1}}{1 - q_A}\frac{1 - q_B^{n+1}}{1 - q_B}\frac{e^{q_A N} - 1}{e^{q_A} - 1} + q_B^{n+1}\right]\|f\|, \quad (5)$$

where N is the smallest natural number such that $q_A = \frac{L}{N} < 1, L$ is Lipschitz coefficient of the operator A, q_B is a contraction coefficient of the operator $B,\ n = 1, 2, \ldots$.

Next, we introduce the following technical well-known theorem for the proof of the main result.

Theorem 2.5. (see [4]) *Assume that, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable mapping and satisfies the following conditions:*
(i) $\frac{\partial f_i}{\partial x_i} \geq \alpha,\ i = \overline{1, n},$
(ii) $\left|\frac{\partial f_i}{\partial x_j}\right| \leq \beta, i \neq j,\ \ i, j = \overline{1, n}.$
Then, the following inequality holds

$$\langle F(x) - F(y), x - y\rangle \geq [\alpha - (n-1)\beta]\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n. \quad (6)$$

Remark 2.6. (see [4]) If $\alpha - (n-1)\beta = 0$ then F is monotone mapping. If $\alpha - (n-1)\beta > 0$ then F is strongly monotone mapping.

3 Main results

In this section, we propose an extension of the PCM for solving perturbed system of nonlinear equations in several variables

$$x + F(x) + \Phi(x) = b, \quad (7)$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is unknown vector, $b = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$ is a given vector, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T, \Phi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^T$ and $F, \Phi : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear mappings.

Theorem 3.1. *Suppose that the following conditions are satisfied*

(i) *F is a differentiable mapping;*

(ii) $\frac{\partial f_i}{\partial x_i} \geq \alpha,\ i = \overline{1, n};$

(iii) $\left|\frac{\partial f_i}{\partial x_j}\right| \leq \beta, i \neq j,\ \ i, j = \overline{1, n};$

(iv) $\alpha - (n-1)\beta \geq 0;$

(v) $\sum\limits_{j=1}^{n} \left|\frac{\partial f_i(x_1, x_2, \ldots, x_n)}{\partial x_j}\right| \leq L, \forall x \in \mathbb{R}^n, \forall i = \overline{1, n};$

(vi) *There is a positive real number $\overline{q} < 1$ such that*

$$\|\Phi(x) - \Phi(y)\| \leq \overline{q}\,\|x - y\|,\ \forall x, y \in \mathbb{R}^n.$$

Then the perturbed system of nonlinear equations (7) *has a unique solution for any $b \in \mathbb{R}^n$.*

Proof. From the assumptions (i)-(iv), by Theorem 2.5 and Remark 2.6 we have F is monotone mapping. Let $F'(x) \equiv A = (a_{ij})$, where $a_{ij} = \frac{\partial f_i(x_1,x_2,...,x_n)}{\partial x_j}$. The assumption (v) is equivalent to $\sum_{j=1}^{n} |a_{ij}| \le L$. For any $h = (h_1, h_2, ..., h_n) \in \mathbb{R}^n$, we have

$$\sum_{j=1}^{n} |a_{ij}h_j| = \sum_{j=1}^{n} |a_{ij}||h_j| \le \sum_{j=1}^{n} |a_{ij}| \max_{1\le j\le n} |h_j| = \sum_{j=1}^{n} |a_{ij}| \|h\|_\infty.$$

It follows that

$$\|A(h)\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}h_j| \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \|h\|_\infty \le L\|h\|_\infty.$$

Hence $\|A\|_\infty \le L < +\infty$ then F is Lipschitz-continuous mapping with Lipschitz coefficient equal to L. It follows from the assumption (vi) that Φ is a contraction mapping with contraction coefficient equal to $\overline{q} < 1$.

Consequently, all conditions of Theorem 2.4 are satisfied. By Theorem 2.4, the perturbed system of nonlinear equations (7) has a unique solution for any $b \in \mathbb{R}^n$. This completes the proof. □

Remark 3.2. In the perturbed system of nonlinear equations (7), we can consider the monotone and Lipschitz-continuous mapping F as the main mapping while the contractive operator Φ as a pertubation mapping or vice versa. This result is the extend of the known result on the application of the method of contractive mapping for solving system of nonlinear equations and result in [4] about PCM for solving system of nonlinear equations. Indeed, we consider the two following special cases. When $F \equiv 0$ the equation (7) has form $x + \Phi(x) = b$ with Φ is a contraction mapping. When $\Phi \equiv 0$ the equation (7) has form $x + F(x) = b$ with F is monotone and Lipschitz - continuous mapping.

By substituting F, Φ, b for A, B, f in iteration processes (4a)–(4d), respectively, the approximate solutions of the perturbed system of nonlinear equations (7) can be found by the following iteration processes

$$x_{i+1} = -\varepsilon_0 F(x_i) + x_j^{(1)},\ i = 0, 1, \dots, \tag{8a}$$

$$x_{j+1}^{(1)} = -\varepsilon_0 F G_1^{-1}(x_j^{(1)}) + x_l^{(2)},\ j = 0, 1, \dots, \tag{8b}$$

$$..., \tag{8c}$$

$$x_{p+1}^{(N)} = -\Phi G_1^{-1} \cdots G_N^{-1}(x_p^{(N)}) + f, p = 0, 1, \dots, \quad x_0^{(N)} = b, \tag{8d}$$

where $x^{(1)} = x + \varepsilon_0 F(x) \equiv G_1(x), x^{(k+1)} = x^{(k)} + \varepsilon_0 F G_1^{-1} \cdots G_k^{-1}(x) \equiv G_{k+1}(x)$, $k = 1, 2, \dots, N-2$.

Assume that, the numbers of steps in each iteration scheme of iteration processes (8a)–(8d) is the same and equals n. Let x_n be approximate solutions of the perturbed system of nonlinear equations (7). Note that x_n depends on N, hence we denote $x(n, N) \equiv x_n$. We have the following theorem.

Theorem 3.3. *Let the assumptions of Theorem 3.1 be satisfied. Then the sequence of approximate solutions* $\{x(n, N)\}, n = 1, 2, \dots$ *constructed by iteration processes* (8a)–(8d) *converges to the exact solution* $x \in \mathbb{R}^n$ *of the perturbed system of nonlinear equations* (7). *Moreover, the following estimates hold*

$$\|x(n, N) - x\| \le \frac{1}{1-\overline{q}} \left[\frac{q^{n+1}}{1-q} \frac{1-\overline{q}^{n+1}}{1-\overline{q}} \frac{e^{qN}-1}{e^q-1} + \overline{q}^{n+1} \right] \|b\|,\ n = 1, 2, \dots, \tag{9}$$

where N *is the smallest natural number such that* $q = \frac{L}{N} < 1, L$ *is Lipschitz coefficient of the operator* F *and* $\overline{q}$ *is a contraction coefficient of the operator* Φ.

Proof. The proof follows immediately from Theorem 2.4 by setting $A = F, B = \Phi$ and $f = b$. □

Example 3.1. Consider the following system of nonlinear equations

$$\begin{cases} \frac{4}{3}x_1 + ln(e^{x_1} + 1) + \frac{1}{3}x_2 + \frac{2}{3}sinx_2 = 1 \\ -\frac{1}{3}x_1 + \frac{3}{4}cosx_1 + \frac{4}{3}x_2 + arctanx_2 = \frac{1}{2} \end{cases}$$

We write this system as the form

$$x + F(x) + \Phi(x) = b,$$

where $x = (x_1, x_2)^T, b = (1 - ln(2), -\frac{1}{4})^T$, $F(x) = (f_1(x_1, x_2), f_2(x_1, x_2))^T$ with

$$f_1(x_1, x_2) = \frac{1}{3}x_1 + ln(e^{x_1} + 1) + \frac{1}{3}x_2 - ln(2),$$

$$f_2(x_1, x_2) = -\frac{1}{3}x_1 + \frac{1}{3}x_2 + arctanx_2,$$

and $\Phi(x) = (\frac{2}{3}sinx_2, \frac{3}{4}cosx_1 - 3/4)^T$.

It is easy to verify that $\frac{\partial f_i}{\partial x_i} \ge \alpha = \frac{1}{3}, i = 1, 2, 3$ and $\left|\frac{\partial f_i}{\partial x_j}\right| = \beta = \frac{1}{3}, i \ne j, \quad i, j = 1, 2$, so that $\alpha - (n-1)\beta = \frac{1}{3} - (2-1)\frac{1}{3} = 0$. Moreover, we have $\sum_{j=1}^{2} |\frac{\partial f_i(x_1,x_2)}{\partial x_j}| \le L = \frac{5}{3}, \forall x = (x_1, x_2) \in \mathbb{R}^2, i = 1, 2.$

For all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ we have

$$\|\Phi(x) - \Phi(y)\| \le \frac{3}{4} \|x - y\|.$$

Therefore, the conditions of Theorem 3.1 are satisfied. Then the given system has a unique solution. By applying the iteration processes (8a)–(8d) and the error estimations (9) with $N = 2$, for some n the approximate solutions of this system and corresponding errors are shown in Table 1.

Table 1. Approximation solutions and errors values in Example 3.1

s	Approximate solutions	Error
20	$(0.2035255893, -0.07148406388)$	1.19371495
30	$(0.2035249334, -0.07148567987)$	$1.928279061 \times 10^{-1}$
50	$(0.2035249230, -0.07148567278)$	$5.025545612 \times 10^{-3}$

4 Conclusions

In this paper, we have presented an extension of the PCM for solving perturbed systems of nonlinear eqations. We first establish the sufficient conditions for the existence and uniqueness of the solution. Then, error analysis of the proposed method is also provided. Finally, a numerical example is given to illustrate the effectiveness of the result.

REFERENCES

[1] Aslam Noor, M., Waseem, M. (2009). *Some iterative methods for solving a system of nonlinear equations*, Comput. Math. Appl. 57: 101—106.

[2] Bernstein, S. N. (1906). *Sur la généralisation du problème de Dirichlet*, Math. Ann. 62: 253–271.

[3] Binh, N. T., Ninh, K. V. (2019). *Parameter continuation method for solving nonlinear Fredholm integral equations of the second kind*, Bull. Malays. Math. Sci. Soc. 42(6): 3379—3407.

[4] Binh, N. T. (2019). *Parameter continuation method for solving systems of nonlinear equations*, Hanoi Pedagogical University 2 J. Sci. 63: 3–13.

[5] Binh, N. T. (2021). *A new extension of parameter continuation method for solving operator equations of the second kind*, Tan Trao University J. Sci. 21: 157–165.

[6] Chuong, N. M., Khai, N.V., Ninh, K.V., Tuan, N.V., Tuong, N. (2000). *Numerical analysis*, VietNam Education Publishing House, Hanoi.

[7] Gaponenko, Y. L. (1986). *The parameter - extension method for an equation of the second kind with a Lipschitz - continuous and monotonic operator*, Comput. Maths. Math. Phys. 26(8): 1123–1131.

[8] Ghane-Kanafi, A., Kordrostami, S. (2016). *A New Approach for Solving Nonlinear Equations by Using of Integer Nonlinear Programming* , Appl. Math. 7: 473-–481.

[9] Leray, J., Schauder, J. (1934). *Topologie et équations fonctionnelles*, Ann. Ec. Norm. Sup. 51: 45–78.

[10] Ninh, K. V. (1999). *Approximate solutions of the equation of a second kind with sum of two operators*, Proc. Inst. Math. Mech. Azerb. Acad. Sci. V(X): 97-–101.

[11] Ninh, K. V. (2011). *A method of extending by parameter for approximate solutions of operator equations*, Acta. Math. Vietnam. 36(1): 119-–127.

[12] Ninh, K. V., Binh, N. T. (2019). *Analytical solution of Volterra–Fredholm integral equations using hybrid of the Method of contractive mapping and Parameter continuation method*, Int. J. Appl. Comput. Math. 5(76): 1–20.

[13] Ortega, J. M., Rheinboldt, W. C. (1970). *Iterative solution of nonlinear equations in several variables*, Academic Press, New York.

[14] Trenogin, V. A. (1980). *Functional Analysis*, Nauka, Moscow.

[15] Trenogin, V. A.(1996). *Locally invertible operator and parameter continuation method. Funktsional*, Anal. i Prilozhen. 30(2): 93–95.

[16] Trenogin, V. A. (1996). *Global invertibility of nonlinear operator and the method of continuation with respect to a parameter*, Dokl. Akad. Nauk. 350(4): 1–3.

[17] Trenogin, V. A. (1998). *Invertibility of nonlinear operators and parameter continuation method (English summary). Spectral and scattering theory, Edited by Alexander G. Ramm*, Plenum Press, New York.

[18] Vetekha, V. G.(2000). *Parameter continuation method for ordinary differential equations*, Proceedings of the 2^{nd} ISAAC Congress, Vol.1, Fukuoka, Japan, August 16—21.