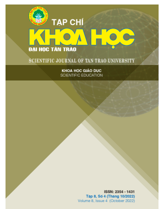
Vol 8. No 4_October 2022

TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

http://tckh.daihoctantrao.edu.vn/



PHƯƠNG PHÁP DIỆN TÍCH Ở BẬC TIỂU HỌC VỚI BÀI TOÁN ĐƯỜNG TRUNG TUYẾN

Trần Đình Tướng[1], *Khổng Chí Nguyện*[2,*]

[1] *Trường Đại học Tài chính - Marketing, thành phố Hồ Chí Minh, Việt Nam*

[2] *Trường Đại học Tân Trào, Tuyên Quang, Việt Nam*

[*]Email: *nguyenkc69@gmail.com*

https://doi.org/10.51453/2354-1431/2022/5xx

Thông tin bài viết

Ngày nhận bài: 21/9/2022
Ngày duyệt đăng: .../.../2022

Từ khóa: *Tam giác, Diện tích, Tỷ số, Đường cao, Cạnh đáy*

Tóm tắt:

Khi giải được một bài toán nói chung và bài toán hình học nói riêng, ta thường có suy nghĩ để tìm thêm lời giải khác, hoặc làm thế nào để có được một bài toán mới từ bài toán gốc. Đối với bài toán hình học, để có thêm những bài toán mới, ta thường thay đổi, hoặc bổ sung thêm giả thiết/kết luận... Trong bài báo này xuất phát từ việc giải Bài toán đường trung tuyến cho học sinh tiểu học, ta sẽ phát triển, mở rộng thêm một số bài toán mới trên cơ sở thay đổi tỷ lệ của mỗi điểm chia trên hai cạnh của tam giác và/hoặc bổ sung thêm những giả thiết phù hợp.

Vol 8. No 4_October 2022

TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

http://tckh.daihoctantrao.edu.vn/

THE AREA METHOD AT THE PRIMARY LEVEL WITH MEDIAN LINE PROBLEMS

Tran Dinh Tuong[1], *Khong Chi Nguyen*[2,*]

[2] *University of Finance - Marketing, Ho Chi Minh City, Viet Nam*

[2] *Tan Trao University, Tuyen Quang, Viet Nam*

[*] Email: *nguyenkc69@gmail.com*

https://doi.org/10.51453/2354-1431/2022/5xx

Article info.

Received: 21/9/2022
Accepted: .../.../2021

Keywords: *Triangle, Area, Ratio, Altitude, Base side*

Abtract:

When solving any problem or geometry problem, we often have thoughts and find the other solutions or how to discover new problems from the original problem. For geometry problems, in order to have a new problem we often change or add assumptions/conclusions and etc. In this work, it comes from solving the median line problem at primary level, we will develop and expand it to include new problems on the basis of changing the ratio of each dividing point on the two sides of the triangle and/or adding appropriate assumptions.

1 Introduction

In the Primary Math program, geometry problems are always content of knowledge with many difficult math problems. In particular, the problems of the area, and the ratio of measurements between geometric figures are challenges for most pupils and some teachers. These problems require students and teachers to have a solid foundation of geometric knowledge, to have coherent thinking and, to be able to read and analyze geometric figures to determine the exact solution for the problem.

In the entrance exams to grade 6 at key junior high schools, the problem of the geometrical area is always taken to increase the differentiation for candidates. However, most pupils have no points in the problems of the area and are solved by the area method. Finding the solution of a geometry problem by the area method and appropriately presenting the object, and helping pupils develop thinking capacity is always a requirement for every teacher.

Starting from the Median line Problem, Example 3.1, we will develop and expand many more problems by changing the ratio of each dividing point on the two sides of the triangle and/or adding make appropriate assumptions. In the scope of this paper, we mainly study area problems related to triangles and special quadrilaterals such as parallelograms, trapezoids, rectangles.

The article is structured as follows: Following the Introduction section is the Preliminary knowledge section, which recalls some related knowledge that will be used in Section 3 of this article. Section 3 is the research content. Finally, the Conclusion section.

2 Preliminary

2.1 Area formulas

Triangle area: Given triangle ABC, altitude AH is perpendicular to side BC at point H. We have, the area formula of triangle ABC is

$$S_{ABC} = \frac{BC \times AH}{2}.$$

BC is called the base side, and AH is the altitude corresponding to BC. The formula for area is (verbally) stated as follows: *The area of a triangle is the product of the base length with the corresponding height divided by* 2.

Now we will consider the ratio of two triangles through the base lengths and the heights.

Consider triangle MNP, altitude MK is perpendicular to NP at point K. We have,

$$S_{MNP} = \frac{NP \times MK}{2}.$$

Then,

$$\frac{S_{ABC}}{S_{MNP}} = \frac{BC \times AH}{NP \times MK} = \frac{BC}{NP} \times \frac{AH}{MK}.$$

If $BC = NP$ then

$$\frac{S_{ABC}}{S_{MNP}} = \frac{AH}{MK}.$$

If $AH = MK$ ten

$$\frac{S_{ABC}}{S_{MNP}} = \frac{BC}{NP}.$$

Area of a trapezoid, parallelogram, and rectangle: Given trapezoid $ABCD$, small base AB, longer base CD, and the altitude AH perpendicular to the bases at H on the side CD. Then, the area of trapezoid $ABCD$ is calculated by the formula

$$S_{ABCD} = \frac{(AB + CD) \times AH}{2}.$$

If $AB = CD$, we get the parallelogram $ABCD$, and the area of parallelogram $ABCD$ is calculated by the formula

$$S_{ABCD} = AB \times AH.$$

If $AB = CD$ and the altitude AH is the side AD, we get the rectangle $ABCD$, and the area of rectangle $ABCD$ is calculated by the formula

$$S_{ABCD} = AB \times AD.$$

2.2 Find two numbers when the sum and difference are known

Let's say $a > b$. Find two numbers a and b if the sum is $T = a + b$, the difference is $H := a - b$. We have

$$a = (T + H) \div 2, \; b = (T - H) \div 2,$$

or $b = T - a$, or $b = a - H$.

2.3 Find two numbers when the sum/difference and ratio are known

Let's say $a > b$. Find two numbers a and b if the sum is $T = a + b$ (or, difference is $H := a - b$) and the ratio is $\frac{a}{b} = \frac{M}{N}$.

It is easy to see that the sum (or, difference) of two numbers a and b is divided into $M + N$ (or, $M - N$) parts, where a occupies the M part, b occupies the N part. Then each part has a value of $p = T \div (M+N)$ (or, $p = H \div (M-N)$), and we have

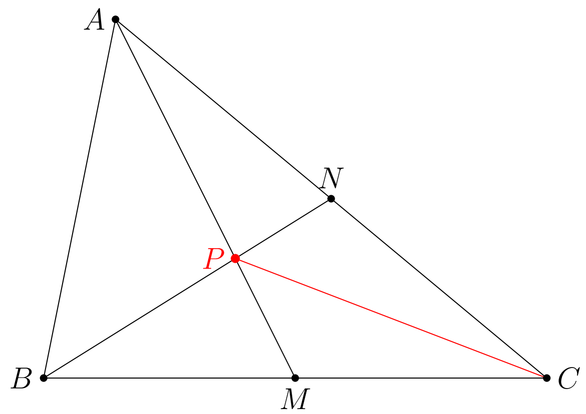
$$a = M \times p,\ b = N \times p.$$

or $b = T - a$ (or, $b = a - H$).

3 Contents

We consider and solve the problem of the median line theorem in a triangle, in line with the thinking of primary students.

Example 3.1. Given triangle ABC. Take the points M, N on the sides BC, CA respectively such that $MC = MB$, $NA = NC$. Let P be the intersection of AM and BN. Prove that $PA = 2 \times PM$, $BP = 2 \times PN$.



ANALYSIS. By assumption, to prove $PA = 2 \times PM$, we will prove $S_{CPA} = 2 \times S_{CPM}$.

Draw a line segment PC, it is easy to see that $S_{PBC} = 2 \times S_{PCM}$, $S_{PBC} = S_{PAC}$. So, $S_{CPA} = 2 \times S_{CPM}$. Therefore,

$$PA = 2 \times PM.$$

Similarly, we also prove the case $BP = 2 \times PN$.

SOLUTION. Draw a line segment PC. By assumption, we have

$$BC = 2 \times BM, \tag{3.1}$$

$$AC = 2 \times AN \tag{3.2}$$

Two triangles PBM and PBC share the same altitude drawn from P to BC and the equality (3.1), so we have

$$S_{PBC} = 2 \times S_{PBM}. \tag{3.3}$$

Two triangles BAN and BCN share the same altitude drawn from B to AC and $NA = NC$, so we have

$$S_{BAN} = S_{BCN}. \tag{3.4}$$

Two triangles PAN and PCN share the same altitude drawn from P to AC and $NA = NC$, so we have

$$S_{PAN} = S_{PCN}. \tag{3.5}$$

By (3.4) and (3.5), we have

$$S_{PAB} = S_{PBC}. \tag{3.6}$$

Therefore, by (3.3) and (3.6), we get

$$S_{PBA} = 2 \times S_{PBM} \Rightarrow \frac{S_{PBA}}{S_{PBM}} = 2. \tag{3.7}$$

Two triangles PBA and PBM share the same altitude drawn from B to AM and the ratio (3.7). Then, the ratio of the bases is equal to the area ratio of the triangles. We have

$$\frac{PA}{PM} = \frac{S_{PBA}}{S_{PBM}} = 2 \Rightarrow PA = 2 \times PM.$$

Next, we prove $BP = 2 \times PN$. It is clear that the two triangles PCN and PAC share the same altitude drawn from P to AC and the ratio (3.2), so we have

$$S_{PAC} = 2 \times S_{PCN}. \tag{3.8}$$

Two triangles ABM and ACM share the same altitude drawn from A to BC and $MB = MC$, so we have

$$S_{ABM} = S_{ACM}. \tag{3.9}$$

Two triangles PBM and PCM share the same altitude drawn from P to BC and $MB = MC$, so we have

$$S_{PBM} = S_{PCM}. \qquad (3.10)$$

By (3.9) and (3.10), we get

$$S_{PAB} = S_{PAC}. \qquad (3.11)$$

By (3.8) and (3.11), we get

$$S_{APB} = 2 \times S_{APN} \Rightarrow \frac{S_{APB}}{S_{APN}} = 2. \qquad (3.12)$$

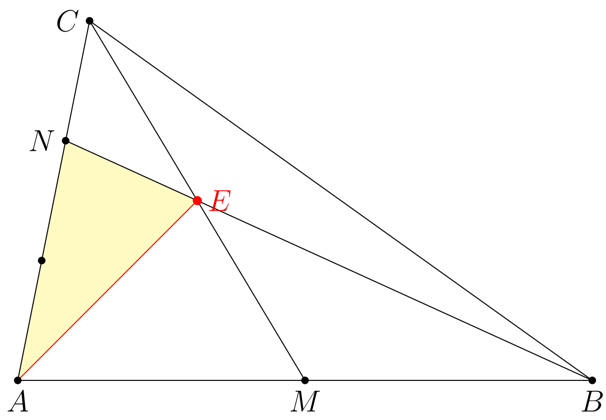
Two triangles APB and APN share the same altitude drawn from A to BN and the ratio (3.12), so the ratio of the two bases is equal to the area ratio of the triangles. We have

$$\frac{PB}{PN} = \frac{S_{APB}}{S_{APN}} = 2 \Rightarrow PB = 2 \times PN.$$

The proof is complete.

Example 3.1 is a basic problem for the requirement to calculate the ratio of the lengths of two line segments on a side connecting the vertex of the triangle to the midpoint of the opposite side by using the area method. To solve the problem, detecting and drawing the subline PC play an important role. To get more new problems, we will change the ratio of the points M, N on the two sides of the triangle ABC and/or add another, appropriate assumptions. We consider some such problems through the examples presented as follows.

Example 3.2. Given triangle ABC. Take the points M on AB, N on AC such that $AM = MB$, $NA = 2 \times CN$. The notation E is the intersection of AM and BN. Suppose, the area of triangle AEN is equal to $18cm^2$. Calculate the area of triangle ABC.



ANALYSIS. Thank to the assumption $NA = 2 \times NC$, we can calculate the area of triangle EAC, the area ratio of the triangles EBC and EBA, namely

$$S_{EAC} = 27cm^2, \ \frac{S_{EBC}}{S_{EBA}} = \frac{1}{2}.$$

On the other hand,from the assumption $AM = MB$, then the area ratio of the triangles EBM and EBA is

$$\frac{S_{EBM}}{S_{EBA}} = \frac{1}{2}.$$

We infer that two triangles EBC and EBM have equal areas and

$$\frac{S_{EBC}}{S_{EBM}} = 1,$$

and then $EC = EM$. Therefore,

$$S_{AEM} = S_{AEC} = 27cm^2.$$

Then, triangle ABC has an area of $108cm^2$.

SOLUTION. By the assumption $NA = 2 \times NC$, we have the ratio $\frac{NA}{AC} = \frac{2}{3}$. Therefore, the area ratio of the triangles ENA and EAC is

$$\frac{S_{ACE}}{S_{ENA}} = \frac{3}{2} \Rightarrow S_{ACE} = 27(cm^2). \qquad (3.13)$$

It is also from the assumption $NA = 2 \times NC$, we have the ratio $\frac{NC}{NA} = \frac{1}{2}$. Thus we have the area ratio of the triangles ENC and ENA as $\frac{S_{ENC}}{S_{ENA}} = \frac{1}{2}$. Thus, two altitudes drawn from C,

A to NB have the ratio $\frac{1}{2}$. So the area ratio of the triangles EBC and EBA is

$$\frac{S_{EBC}}{S_{EBA}} = \frac{1}{2} \Rightarrow S_{EBC} = \frac{1}{2} \times S_{EBA}. \quad (3.14)$$

On the other hand, under the assumption $AM = MB$, we have $AB = 2 \times MB$. Therefore, the area ratio of the triangles EMB and EBA is

$$\frac{S_{EMB}}{S_{EBA}} = \frac{1}{2} \Rightarrow S_{EAB} = 2 \times S_{EMB}. \quad (3.15)$$

By (3.14) and (3.15), we have $S_{BEC} = S_{BEM}$. Since two triangles BEC and BEM share the same altitude drawn from B to MC, so two bases corresponding to B must be equal to each other, and $EC = EM$. We get

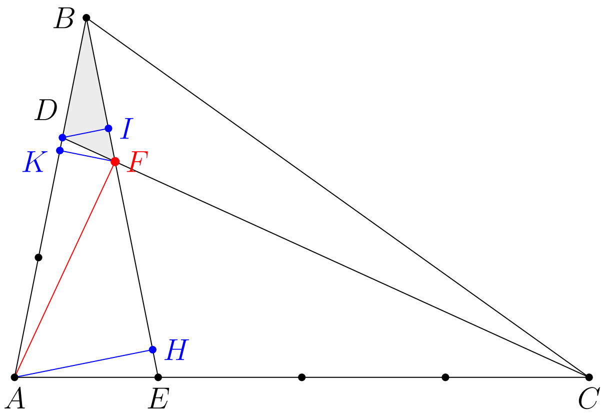
$$S_{EMA} = S_{ECA} = 27(cm^2).$$

Thus, we have $S_{CAM} = 2 \times S_{EAM} = 54(cm^2)$. So, $S_{ABC} = 2 \times S_{CAM} = 108cm^2$.

Answer: $S_{ABC} = 108cm^2$

In Example 3.2, the "subline" AE is given. Then, to get the solution of the problem, we have to calculate the ratio $\frac{EC}{EM} = 1$, or the ratio $\frac{EB}{EN} = 3$. This is the critical middle step to determine S_{ABC}. Therefore, this problem may require proving $EC = EM$ or $EB = 3 \times EN$ to get another problem. We can also replace the area of triangle ENA with the area of triangle ENC, EAM and etc. Finally, F is the intersection of the extended AE that intersects the side BC. It is easy to see that the ratio $\frac{AE}{AF} = \frac{BE}{BN} = \frac{3}{4}$. Readers should prove these claims. In the next example, we add more assumptions in addition to the ones about the dividing point on two sides of the triangle.

Example 3.3. Given triangle ABC. Take the points D on AB, E on AC such that $AB = 3 \times BD$, $AC = 4 \times AE$. The notation F is the intersection of CD and BE. Suppose, the area of triangle DBF is equal to $100cm^2$. Calculate the area of triangle ABC.



ANALYSIS. By assumption, it is easy to see that the triangle DBF lies within the triangle ABE and $S_{ABC} = 4 \times S_{ABE}$. Therefore, to compute S_{ABC}, we would have to find S_{ABE} through S_{DBF}. Two triangles DBF, ABE have two bases on the side BE, so the altitudes are parallel. Draw the altitude AH, DI perpendicular to BE. We will calculate the ratio of two altitudes and two bases of the triangles DBF, ABE. Thereby, the ratio of the areas of the two triangles DBF, ABE is

$$\frac{S_{ABE}}{S_{DBF}} = \frac{AH}{DI} \times \frac{BE}{BF} = \frac{15}{2}.$$

So $S_{ABE} = 750cm^2$. Hence $S_{ABC} = 3000cm^2$.

SOLUTION. Draw AF and the altitudes AH, DI perpendicular to BE, H, I are in AE. We have

$$S_{ABE} = \frac{BE \times AH}{2}, S_{DBF} = \frac{BF \times DI}{2}. \quad (3.16)$$

i) Computing the ratio $\frac{AH}{DI}$. Draw AF, and FK perpendicular to AB. The triangle FDB and the triangle FAB share the same altitude FK, and the ratio of the bases $\frac{AB}{DB} = 3$ (by assumption $AB = 3 \times BD$). So $\frac{S_{FAB}}{S_{FDB}} = 3$. Hence,

$$S_{FAB} = 3 \times S_{FDB}. \quad (3.17)$$

Now consider two triangles DFB and AFB with the base side FB. Combined with (3.17) we get the ratio of two altitudes $\frac{AH}{DI} = 3$. We deduce,

$$AH = 3 \times DI. \quad (3.18)$$

ii) Computing the ratio $\frac{BE}{BF}$ through the ratio area $\frac{S_{CFE}}{S_{CFB}}$. Two triangles CAD and CBD share the same altitude drawn from C to AB and the ratio of the two bases $\frac{AD}{BD} = 2$ (since $AB = 3 \times BD$), so the area ratio is

$$\frac{S_{CAD}}{S_{CBD}} = 2.$$

We deduce

$$S_{CAD} = 2 \times S_{CBD}. \quad (3.19)$$

Now we consider two triangles CAD, CBD with base CD. Combined with (3.19), we get the ratio of two altitudes drawn from A and B to CD equal to 2.

The triangles CFA and CFB share the base CF, and the ratio of two altitudes corresponding to the base CF is 2. Hence the area ratio is $\frac{S_{CFA}}{S_{CFB}} = 2$, so

$$S_{CFB} = \frac{1}{2} \times S_{CFA}. \quad (3.20)$$

Two triangles FAC, FEC share the altitude drawn from F to AC and the ratio of two bases is $\frac{AC}{EC} = \frac{4}{3}$ (by assumption, $AC = 4 \times AE$). So, the area ratio of two the triangles is $\frac{S_{FAC}}{S_{FEC}} = \frac{4}{3}$. We get

$$S_{FAC} = \frac{3}{4} \times S_{FEC}. \quad (3.21)$$

It thanks to (3.20), (3.21), we have $S_{CFB} = \frac{1}{2} \times \frac{4}{3} \times S_{CFE} = \frac{2}{3} \times S_{CFE}$. Hence the area ratio is

$$\frac{S_{CFE}}{S_{CFB}} = \frac{3}{2}. \quad (3.22)$$

According to the results (3.22), the two triangles CFE and CFB share the altitude drawn from C to BE, so the ratio of two bases is $\frac{FE}{FB} = \frac{3}{2}$. Thus, we have

$$\frac{BE}{FB} = \frac{FE + FB}{FB} = \frac{3+2}{2} = \frac{5}{2}.$$

We deduce

$$BE = \frac{5}{2} \times FB. \quad (3.23)$$

iii) Computing the area of triangle ABC**.** By the (3.16), (3.18) and (3.23) we get

$$S_{ABE} = \frac{1}{2} \times BE \times AH = \frac{15}{2} \times \frac{1}{2} \times BF \times DI$$
$$= \frac{15}{2} \times S_{DBF} = \frac{15}{2} \times 100 = 750(cm^2).$$

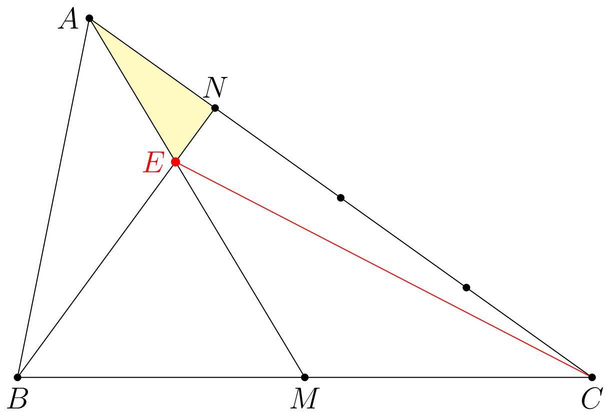
Two triangles ABE, ABC share the altitude drawn from B to AC, $AC = 4 \times AE$. Thus,

$$S_{ABC} = 4 \times S_{ABE} = 4 \times 750 = 3000(cm^2).$$

Answer: $S_{ABC} = 3.000cm^2$

Example 3.3 has the solution presented with a different approach, through finding the ratio between the altitudes, the bases of two triangles DBF and ABE. So we determine the area ratio of these triangles and the area of the triangle ABF - an important calculation in computing the area of the triangle ABC. Of course, we can also compute the area of triangle ABC through the area of triangle BCD.

Example 3.4. Given triangle ABC. Take the point M in BC, the point N in AC such that, $BM = MC$, $CN = 3 \times NA$. E is the intersection of AM and BN. If the area of triangle ABC is equal to $420cm^2$, calculate the area of triangle AEN and the ratio $\frac{AE}{AM}$.



ANALYSIS. By the assumptions and conditions about two triangles having equal area, it is easy to see that

$$S_{AEN} = \frac{1}{4} \times S_{AEC} = \frac{1}{4} \times S_{AEB}.$$

Hence, $S_{AEN} = \frac{1}{5} \times S_{ABN}$. We deduce

$$S_{AEN} = \frac{1}{20} \times S_{ABC} = 21(cm^2).$$

SOLUTION. Draw the line segment EC. By assumption, triangle ABM and triangle AMC share the altitude drawn from A to BC and $BM = MC$. Therefore, $S_{ABM} = S_{ACM}$.

Similarly, triangle EBM and triangle EMC share the altitude drawn from E to BC and $BM = MC$. Hence, $S_{EBM} = S_{ECM}$. Deduce

$$\begin{aligned} S_{ABE} &= S_{ABM} - S_{EBM} \\ &= S_{ACM} - S_{ECM} = S_{ACE} \end{aligned} \quad (3.24)$$

Now we will calculate S_{AEN} in terms of S_{ABN}. By the assumption $CN = 3 \times NA$, then $NA = \frac{1}{4} \times AC$. The triangles AEN and AEC share the altitude drawn from E to AC, and the ratio of two bases is $NA = \frac{1}{4} \times AC$. Therefore

$$S_{AEN} = \frac{1}{4} \times S_{AEC}.$$

Combined with (3.24), we have $S_{AEN} = \frac{1}{4} \times S_{AEB}$, and then

$$S_{AEN} = \frac{1}{5} \times S_{ABN}$$

On the other hand, the triangles ABN and ABC share the altitude drawn from B to BC and the ratio of the bases is $NA = \frac{1}{4} \times AC$, so

$$S_{ABN} = \frac{1}{4} \times S_{ABC} = \frac{1}{4} \times 420 = 105(cm^2).$$

Therefore $S_{AEN} = \frac{1}{5} \times 105 = 21(cm^2)$.

Finally, we calculate the ratio $\frac{AE}{AM}$. The triangles ABE and ABM share the altitude drawn from B to AM. Therefore, the area ratio of the triangles ABE and ABM is equal to the ratio of the bases AE, AM. Therefore,

$$\frac{AE}{AM} = \frac{S_{ABE}}{S_{ABM}}.$$

We have $S_{ABE} = S_{ABN} - S_{AEN} = 105 - 21 = 84(cm^2)$. So, the ratio to be found is

$$\frac{AE}{AM} = \frac{84}{210} = \frac{2}{5}.$$

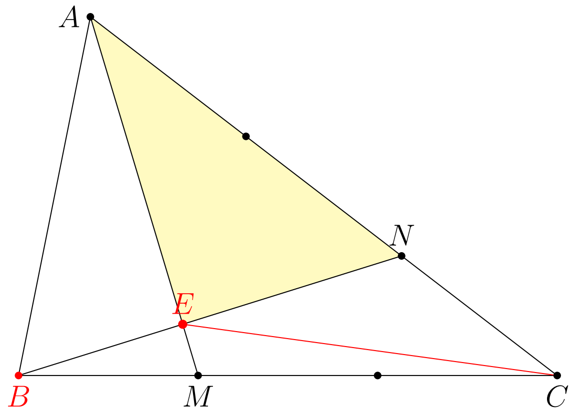
Answer: $S_{AEN} = 21cm^2$; $\frac{AE}{AM} = \frac{2}{5}$.

The subline CE in Example 3.4 is still the deciding factor to get the correct answer. The problem can be changed the assumptions, conclusions, and others to have new problems:

- Change the conclusions about calculating areas and ratios, for example, the area of the triangle AEB or BEM, the ratios $\frac{AE}{EM}$ or $\frac{BE}{BN}$, or $\frac{EN}{EB}$, etc.

- Change the assumption about the position of the points M, N in BC, AC respectively, for example, $BM = \frac{1}{2} \times MC$ and $EN = \frac{1}{2} \times NC$ and etc. These changes can make the more difficult problems. We will consider an example for these cases.

Example 3.5. Given triangle ABC. Take the point M in BC, the point N in AC such that, $BM = \frac{1}{2} \times MC$, $CN = \frac{1}{2} \times NA$, E is the intersection of AM and BN. Supopose that the area of the triangle ABC is $420cm^2$. Calculate the area of the triangle AEN and the ratio $\frac{AE}{AM}$.



ANALYSIS. According to the assumptions and conditions about the area ratio of two triangles with equal altitudes, the area ratio is the ratio of two bases. We have

$$S_{AEN} = \frac{2}{3} \times S_{AEC} = \frac{4}{3} \times S_{AEB}.$$

Therefore, $S_{AEN} = \frac{4}{7} \times S_{ABN}$. We deduce $S_{AEN} = \frac{8}{21} \times S_{ABC} = \frac{8}{21} \times 420 = 160(cm^2)$.

SOLUTION. Draw the line segment EC. By assumption, the triangles ABM and AMC share the altitude drawn from A to BC, and $BM = \frac{1}{2} \times MC$. Therefore, $S_{ABM} = \frac{1}{2} \times S_{ACM}$. Similarly, two triangles EBM and EMC share the altitude from E to BC and $BM = \frac{1}{2} \times MC$. Thus, $S_{EBM} = \frac{1}{2} \times S_{ECM}$. We deduce

$$\begin{aligned} S_{ABE} &= S_{ABM} - S_{EBM} \\ &= \frac{1}{2} \times S_{ACM} - \frac{1}{2} \times S_{ECM} = \frac{1}{2} \times S_{ACE}. \end{aligned}$$

Therefore,

$$S_{ACE} = 2 \times S_{ABE} \qquad (3.25)$$

Now we will calculate the area S_{AEN} through S_{ABN}.

The triangles AEN and AEC share the altitude drawn from E to AC and $NA = \frac{2}{3} \times AC$. So we have

$$S_{AEN} = \frac{2}{3} \times S_{AEC}.$$

Combined with (3.25), we have $S_{AEN} = \frac{4}{3} \times S_{AEB}$. We deduce

$$S_{AEN} = \frac{4}{7} \times S_{ABN}$$

On the other hand, the triangles ABN and ABC share the altitude drawn from B to AC and $NA = \frac{2}{3} \times AC$, then

$$S_{ABN} = \frac{2}{3} \times S_{ABC} = \frac{2}{3} \times 420 = 280(cm^2).$$

Therefore, $S_{AEN} = \frac{4}{7} \times 280 = 160(cm^2)$.

Finally, we calculate the ratio $\frac{EM}{AE}$. Triangles BEM and BEA share the altitude drawn from B to AM. Thus, the area ratio of the triangles BEM and BEA is the ratio of the bases AE, EM. Therefore,

$$\frac{EM}{AE} = \frac{S_{BEM}}{S_{BEA}}.$$

We have

$$S_{BEA} = S_{ABN} - S_{AEN} = 280 - 160 = 120(cm^2),$$

so

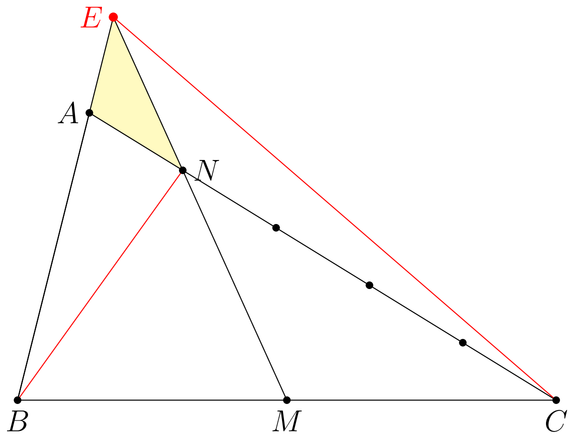
$$S_{BEM} = S_{ABM} - S_{EBA} = 140 - 120 = 20(cm^2).$$

So, the ratio to be found is

$$\frac{EM}{EA} = \frac{20}{120} = \frac{1}{6}.$$

Answer: $S_{AEN} = 160cm^2$; $\frac{EM}{EA} = \frac{1}{6}$.

Example 3.6. Given triangle ABC. Take the points M in BC and N in AC such that $BM = MC$, $CN = 4 \times NA$. The line passing through M and N intersects the side BA extended at E. Determine the area ratio of the triangles ANE, ABC and the length ratio of the sides EN, EM.



ANALYSIS. By the assumptions we have $\frac{S_{ABN}}{S_{ABC}} = \frac{1}{5}$. Therefore, to determine the ratio $\frac{S_{AEN}}{S_{ABC}}$ we must calculate the ratio $\frac{S_{AEN}}{S_{ABN}}$.

It is clear that $S_{EBM} = S_{ECM}$, $S_{NBM} = S_{NCM}$. Therefore, $S_{EBN} = S_{ECN}$. On the other hand, $\frac{S_{EAN}}{S_{ECN}} = \frac{1}{4}$. Hence, $\frac{S_{EAN}}{S_{EBN}} = \frac{1}{4}$, so $\frac{S_{EAN}}{S_{ABN}} = \frac{1}{3}$. We deduce

$$\frac{S_{AEN}}{S_{ABC}} = \frac{1}{15} \text{ and } \frac{EN}{EM} = \frac{2}{5}.$$

SOLUTION. We draw BN and CE. Acording to the assumptions, the triangles EBM and ECM share the altitude drawn from E to BC and $MB = MC$, so $S_{EBM} = S_{ECM}$.

Similarly, we also have $S_{NBM} = S_{NCM}$. We deduce, $S_{EBN} = S_{ECN}$

Two triangles EBM and ECM share the altitude drawn from E to AC and $CN = 4 \times AN$, so we get

$$S_{EAN} = \frac{1}{4} \times S_{ECN}.$$

Therefore,

$$S_{EAN} = \frac{1}{4} \times S_{EBN}.$$

We deduce

$$S_{EAN} = \frac{1}{3} \times S_{ABN}. \quad (3.26)$$

From the assumption $CN = 4 \times AN$, so $AC = 5 \times AN$. Furthermore, the two triangles ABN and ABC share the same altitude from B to AC and $AC = 5 \times AN$, so we have

$$S_{ABN} = \frac{1}{5} \times S_{ABC}. \quad (3.27)$$

Combined with (3.26) we get the result $S_{EAN} = \frac{1}{15} \times S_{ABC}$, or

$$\frac{S_{EAN}}{S_{ABC}} = \frac{1}{15}.$$

Next, we calculate the ratio $\frac{EN}{EM}$. According to (3.26), we have

$$S_{BEN} = \frac{4}{3} \times S_{BNA}. \quad (3.28)$$

Now we will determine S_{BNA} through S_{BNM}. The triangles BNM, BNC share the altitude drawn from the vertex N and $BM = \frac{1}{2} \times BC$, so we have

$$S_{BNM} = \frac{1}{2} \times S_{BNC}.$$

On the other hand, by the assumptions we have $S_{BNC} = \frac{4}{5} \times S_{ABC}$, so $S_{BNM} = \frac{2}{5} \times S_{ABC}$, or

$$S_{ABC} = \frac{5}{2} \times S_{BNM}.$$

Combining with (3.27), we get the result

$$S_{BNA} = \frac{1}{2} \times S_{BNM}.$$

Therefore, by (3.28), we have

$$S_{BNE} = \frac{2}{3} \times S_{BNM}.$$

The trangles BNE, BNM share the altitude drawn from B to EM and the area ratio

$$\frac{S_{BEN}}{S_{BNM}} = \frac{2}{3}.$$

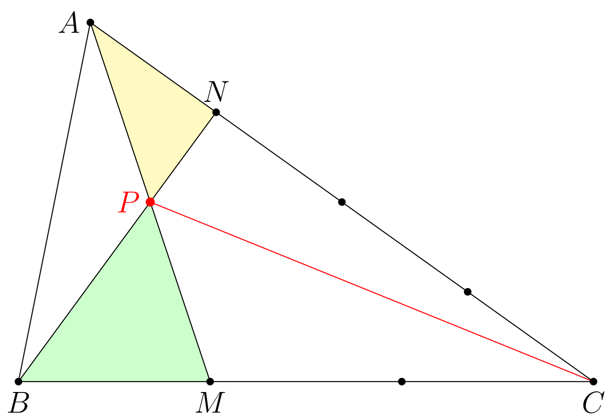
Thus, the ratio of the bases EN, EM is the ratio of two areas S_{BEN}, S_{BNM}, and we have

$$\frac{EN}{NM} = \frac{S_{BEN}}{S_{BNM}} = \frac{2}{3} \Rightarrow \frac{EN}{EM} = \frac{2}{5}.$$

Answer: $\frac{S_{EAN}}{S_{ABC}} = \frac{1}{15}$, $\frac{EN}{EM} = \frac{2}{5}$.

This is a difficult problem that is not given specific data, especially the requirement to find the ratio $\frac{EN}{EM}$. The problem will be more difficult if it only requires calculating the ratio $\frac{EN}{EM}$ or changing the assumption about the point M, for example, $BM = 2 \times CM$. Calculating the ratio $\frac{S_{AEN}}{S_{ABC}}$ is an intermediate step and also requires a lesser degree of difficulty. When adding specific data to simplify the problem, we should start from the area of triangle AEN, or the area of triangle ABC and calculate the area of the remaining triangles. Note that, if we choose a specific figure for the area of triangle ABC, it should be divisible by $3, 4, 5$ so that the results are integers.

Example 3.7. Given triangle ABC. Take the points M in BC and N in AC such that $BM = \frac{1}{3} \times BC$, $AN = \frac{1}{4} \times AC$. The notation P is the intersection of AM and BN. Calculate the area ratio of the triangles PBM, PAN and prove that $PA = PM$.



ANALYSIS. Draw the line segment PC. Thanks to the assumptions we have

$$S_{PBM} = \frac{1}{3} \times S_{PBC} = S_{PAB} \Rightarrow \ PA = PM.$$

On the other hand, it is easy to see that

$$S_{PAB} = \frac{1}{2} \times S_{PAC}, S_{PAN} = \frac{1}{4} \times S_{PAC}.$$

Therefore, $S_{PAB} = 2 \times S_{PAN}$. We deduce $S_{PBM} = 2 \times S_{PAN}$. To solve this problem we will prove $PA = PM$, thereby determining the ratio of two triangles S_{PBM}, S_{PAN}.

SOLUTION. Drawn the line segment PC. By the assumptions, we have

$$AN = \frac{1}{3} \times NC. \quad (3.29)$$

The triangles PBM, PCM share the altitude drawn from P to BC, and $BM = \frac{1}{3} \times BC$, so we have

$$S_{PBM} = \frac{1}{3} \times S_{PBC}. \quad (3.30)$$

The triangles PAN, PCN share the altitude drawn from P to AC, and the ratio (3.29), so we have

$$S_{PAN} = \frac{1}{3} \times S_{PCN}. \quad (3.31)$$

The triangles BAN, BCN share the altitude drawn from P to AC, and the ratio (3.29), so we get

$$S_{BAN} = \frac{1}{3} \times S_{BCN}. \quad (3.32)$$

Acording to (3.31) and (3.32), we have

$$S_{PAB} = \frac{1}{3} \times S_{PBC}. \quad (3.33)$$

Combined with the (3.29), we get the result

$$S_{PAB} = S_{PMB}. \quad (3.34)$$

The triangles BPA, BPM share the altitude drawn from B to AM. Hence, we have immediately

$$PA = PM.$$

Next, we calculate the area ratio of the triangles PAN, PBM.

It is assumed that $BM = \frac{1}{3} \times BC$. Therefore, we have $BM = \frac{1}{2} \times CM$.

The triangles ABM, ACM share the altitude drawn from A to BC, and $BM = \frac{1}{2} \times CM$, so we have

$$S_{ABM} = \frac{1}{2} \times S_{ACM}. \quad (3.35)$$

From (3.29) and (3.35), we deduce

$$S_{PAB} = \frac{1}{2} \times S_{PAC}. \quad (3.36)$$

On the other hand, two triangles PAN, PAC share the altitude drawn from A to AC, and $AN = \frac{1}{4}AC$, so we have

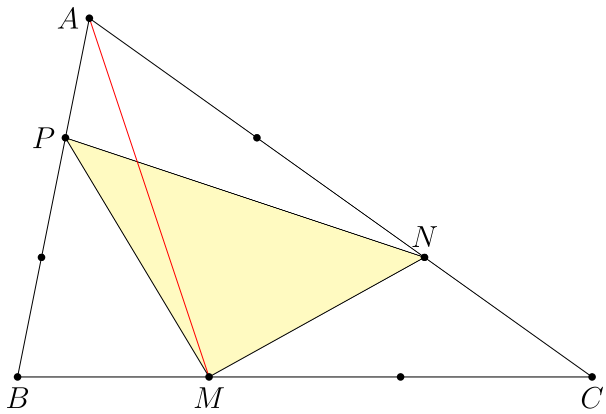
$$S_{PAN} = \frac{1}{4} \times S_{PAC}. \quad (3.37)$$

From (3.34), (3.36) and (3.37), we deduce

$$S_{PAN} = \frac{1}{2} \times S_{PBM} \Rightarrow \frac{S_{PAN}}{S_{PBM}} = \frac{1}{2}.$$

Answer: $\frac{S_{PAN}}{S_{PBM}} = \frac{1}{2}, PA = PM.$

Example 3.8. Given triangle ABC. Take the points M, N, P on the sides BC, CA, AB respectively such that $MC = 2 \times MB, NA = 2 \times NC, PB = 2 \times PA$. Prove that $S_{MNP} = \frac{1}{3} \times S_{ABC}$.



ANALYSIS. Draw the line segment AM. By the assumptions we have $S_{ABM} = \frac{1}{3} \times S_{ABC}$ and $S_{MPB} = \frac{2}{3} \times S_{MBA}$. Therefore,

$$S_{MPB} = \frac{2}{9} \times S_{ABC}.$$

Similarly, we get

$$S_{PNA} = \frac{2}{9} \times S_{ABC}, \ S_{MNC} = \frac{2}{9} \times S_{ABC}.$$

We deduce, $S_{MNP} = \frac{1}{3} \times S_{ABC}$.

SOLUTION. Draw the line segment AM. Acording to the assumptions, we have

$$MB = \frac{1}{3} \times BC, \quad (3.38)$$

$$PB = \frac{1}{3} \times AB. \quad (3.39)$$

Two triangles ABM, ACM share the altitude drawn from A to BC, and the ratio (3.38). We have

$$S_{ABM} = \frac{1}{3} \times S_{ABC}. \quad (3.40)$$

Two triangles MPB, MAB share the altitude drawn from M to AB, and the ratio (3.39). We get

$$S_{MPB} = \frac{2}{3} \times S_{MAB}. \quad (3.41)$$

From (3.40) and (3.41) we get the result

$$S_{MPB} = \frac{2}{9} \times S_{ABC}. \quad (3.42)$$

Similarly, we have

$$S_{MNC} = \frac{2}{9} \times S_{ABC}, \quad (3.43)$$

$$S_{NPA} = \frac{2}{9} \times S_{ABC}. \quad (3.44)$$

Combined the (3.42), (3.43) and (3.44), we deduce

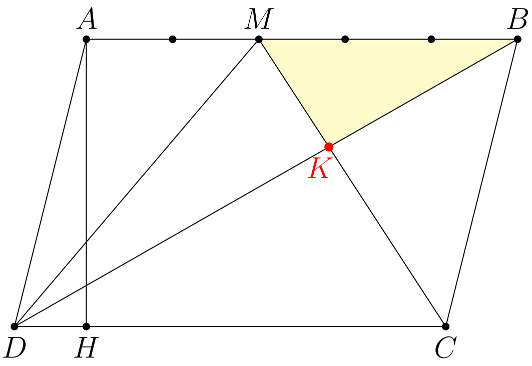
$$S_{MNP} = \frac{1}{3} \times S_{ABC}.$$

The proof is complete.

Example 3.8 is also a basic problem, has a simple solution, and is solved by the area method. However, if the subline AM is not detected, the problem will not be possible to solve. Our problem will becomes more difficult if the points M, N, and P are defined on the sides of the triangle with different ratios.

Next, we will consider some area problems in the special quadrilaterals.

Example 3.9. Given a parallelogram $ABCD$, height $AH = 15cm$, base $AB = 20cm$. Take the point M on the side AB such that $AM = \frac{2}{3} \times MB$. The line segment MC intersects the diagonal BD at K. Calculate the area of triangle KMB.



ANALYSIS. By assumption, it is easy to calculate the area of triangles MBD, MCD, CBD and CBM. On the other hand, assumptively, we have the ratio $\frac{MB}{CD} = \frac{3}{5}$. Therefore, to calculate S_{MBK}, we have to calculate the ratio $\frac{S_{MBK}}{S_{MBD}} = \frac{3}{8}$, or the ratio $\frac{S_{BMK}}{S_{BMC}} = \frac{3}{8}$. Then,

$$S_{BMK} = \frac{3}{8} \times S_{BMD} = \frac{3}{8} \times S_{BMC}.$$

SOLUTION. Acording to the assumptions, we have $MB = \frac{3}{5} \times AB = \frac{3}{5} \times 20 = 12(cm)$. Therefore, we get $S_{DMB} = \frac{1}{2} \times MB \times AH = \frac{1}{2} \times 12 \times 15 = 90(cm^2) = S_{CMB}$,
By the same assumption, we have

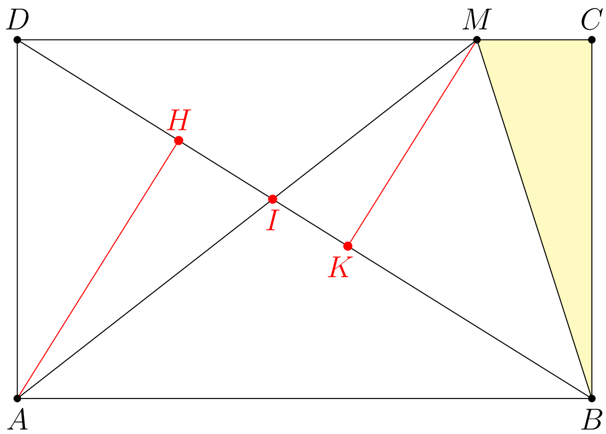
$$S_{MCD} = \frac{1}{2} \times CD \times AH = 150(cm^2).$$

Two triangles MCD, CMB share the altitude $AH = 15(cm)$, and the ratio of the bases $\frac{MB}{CD} = \frac{3}{5}$. Thus, the ratio of the areas is $\frac{S_{CMB}}{S_{MCD}} = \frac{3}{5}$. On the other hand, triangles MCD, CMB share the bases MC. Then, the altitude drawn from the vertices B, D to MC has the ratio $\frac{3}{5}$. The triangles BKM, DKM share the bases KM, and the ratio of the altitudes is $\frac{3}{5}$. So the ratio of the areas is $\frac{S_{BKM}}{S_{DKM}} = \frac{3}{5}$. It is clear that we get the problem: Find two numbers when the sum and ratio of the two numbers are known, where the sum is $S_{BKM} + S_{DKM} = S_{MBD} = 90(cm^2)$, the ratio is $\frac{S_{BKM}}{S_{DKM}} = \frac{3}{5}$ and S_{KMB} is small number. Therefore, we have

$$S_{KMB} = 90 \div (3+5) \times 3 = 33,75(cm^2).$$

Answer: 33,75cm^2.

Example 3.10. Given rectangle $ABCD$. Take the point M in CD, the diagonal BD and the line segment AM intersect at I such that the area of triangle BMC is equal to $36cm^2$ and equal to $\frac{9}{16}$ area of triangle IMD. Calculate the area of rectangle $ABCD$.



ANALYSIS. Acording to the assumptions

$$S_{BMC} = 36cm^2 = \frac{9}{16} \times S_{IMD}.$$

Thus, $S_{IMD} = 64cm^2$, and we find the area of triangle IAB. So, to calculate the area of rectangle $ABCD$ we will compute the area of two triangles IAD, IBM.

Since $S_{IAD} = \frac{1}{2} \times ID \times AH$ and $S_{MID} = \frac{1}{2} \times ID \times MK = 64cm^2$, we would determine the relationship between the line segments ID, IB or AH, MK. Notice that $\frac{ID}{IB} = \frac{MK}{AH}$.

SOLUTION. By assumptions, we have immediately $S_{IMD} = 64cm^2$. Next, the triangles

DAB, MAB share the base side AB and the height DA, so

$$S_{MAB} = S_{DAB} = S_{BCD}. \quad (3.45)$$

On the other hand, the triangles BCD, MAB share the triangles IMB. We deduce

$$S_{IAB} = S_{IDM} + S_{MBC} = 100cm^2.$$

We determine the triangles IAD, IMB. By (3.45) and the triangles DAB, MAB share the triangle IAB, so we have $S_{IAD} = S_{IBM}$, and then

$$\frac{ID}{IB} = \frac{MK}{AH}. \quad (3.46)$$

We drawn the altitudes AH, MK with H, K in BD. Then, we have

$$S_{IMD} = \frac{1}{2} \times ID \times MK, \ S_{IAB} = \frac{1}{2} \times IB \times AH.$$

Therefore,

$$\frac{64}{100} = \frac{S_{IMD}}{S_{IAB}} = \frac{ID}{IB} \times \frac{MK}{AH}. \quad (3.47)$$

By (3.46) and (3.47) we get the result

$$\frac{ID}{IB} \times \frac{ID}{IB} = \frac{64}{100} \Rightarrow \frac{ID}{IB} = \frac{4}{5}. \quad (3.48)$$

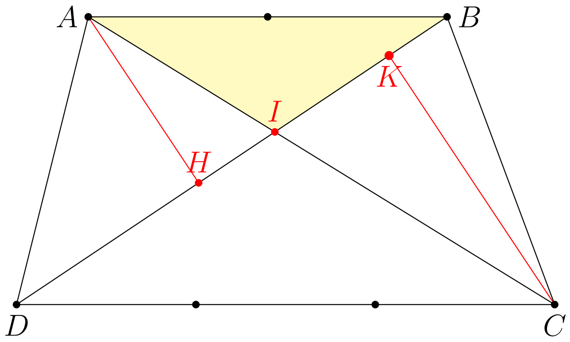
We deduce,

$$S_{IAD} = \frac{1}{2} \times ID \times AH = \frac{1}{2} \times \frac{4}{5} \times IB \times AH$$
$$= \frac{2}{5} \times S_{IAB} = \frac{2}{5} \times 100 = 40(cm^2).$$

From the above results, it follows that $S_{ABCD} = 2 \times S_{ABD} = 2 \times (S_{IAB} + S_{IAD}) = 280cm^2$.

Answer: $S_{ABCD} = 280cm^2$.

Example 3.11 (Junior high school Le Quy Đon, 2020)**.** In a trapezoid $ABCD$, the ratio between the base sides AB, CD is $\frac{2}{3}$. Two diagonals AC, BD intersect at I. The area of triangle AOB is $4cm^2$. Find the area of trapezoid $ABCD$.



ANALYSIS. From the assumptions, it is easy to calculate the area ratio of the triangles DAB and BCD,

$$\frac{S_{DAB}}{S_{BCD}} = \frac{2}{3}.$$

We are also easy to calculate the area ratio of the triangles ABI and CBI,

$$\frac{S_{ABI}}{S_{CBI}} = \frac{2}{3} \Rightarrow S_{CBI} = 6cm^2.$$

Hence, $S_{ABD} = 10cm^2$. So, $S_{BCD} = 15cm^2$, and thereby $S_{ABCD} = 25cm^2$.

SOLUTION. The triangles DAB, BCD have the same height as the height of the trapezoid and the ratio of the bases $\frac{AB}{CD} = \frac{2}{3}$. So the area ratio of two triangles DAB, BCD is

$$\frac{S_{DAB}}{S_{BCD}} = \frac{2}{3}, \text{ hay } S_{BCD} = \frac{3}{2} \times S_{DAB}. \quad (3.49)$$

The triangles ABD, CBD share the base BD and the ratio (3.49). Therefore, the ratio of the altitudes drawn from A and C to BD is $\frac{2}{3}$.

The triangles ABI, CBI share the base BI and the ratio of the altitudes drawn from A and C to BI is $\frac{2}{3}$. We deduce

$$S_{CBI} = \frac{3}{2} \times S_{ABI} = 6cm^2. \quad (3.50)$$

On the other hand, $S_{IAD} = S_{IBC} = 6cm^2$. We deduce,

$$S_{ABD} = S_{IAB} + S_{IAD} = 4 + 6 = 10(cm^2).$$

According to (3.49), we get

$$S_{CBD} = \frac{3}{2} \times 10 = 15(cm^2).$$

So, the area of trapezoid $ABCD$ is

$$S_{ABCD} = S_{ABD} + S_{CBD} = 10 + 15 = 25(cm^2).$$

Answer: $S_{ABCD} = 25cm^2$.

4 Conclusion

The article has presented some development directions to detect new problems such as: changing the assumptions and/or adding new assumptions, changing the role between the assumptions and conclusion, and etc. Improving the solutions to these examples is left to the reader. Otherwise, these techniques that we proposed in this work can be applied to higher levels such as high school and undergraduate programs for some fields, such as geometry in algebra and geometric data analysis. These problems arise in these fields, and some applications of geometry will be studied in future research.

REFERENCES

[1] Bộ Giáo dục và Đào tạo (2020), Toán lớp 4 (Tập 1, 2), Nhà xuất bản Giáo dục Việt Nam.

[2] Bộ Giáo dục và Đào tạo (2020), Toán lớp 5 (Tập 1, 2), Nhà xuất bản Giáo dục Việt Nam.

[3] Đỗ Đình Hoan (Chủ biên), Nguyễn Áng, Đỗ Tiến Đạt, Phan Thanh Tâm (2020), Bài tập Toán 4, Nhà xuất bản Giáo dục Việt Nam.

[4] Đỗ Đình Hoan (Chủ biên), Nguyễn Áng, Đỗ Tiến Đạt, Phan Thanh Tâm (2020), Bài tập Toán 5, Nhà xuất bản Giáo dục Việt Nam.

[5] Trần Phương, Nguyễn Đức Tấn, Phạm Xuân Tiến (2011), Toán Chọn lọc Tiểu học (Tập 2: Các đề toán), Nhà xuất bản Giáo dục Việt Nam.

[1] Ministry of Education and Training, Grade 4 Math (Volume 1, 2), Vietnam Education Publishing House.

[2] Ministry of Education and Training, Grade 5 Math (Volume 1, 2), Vietnam Education Publishing House.

[3] Do Dinh Hoan (Editor), Nguyen Ang, Do Tien Dat, Phan Thanh Tam (2020), Math Exercise 4, Vietnam Education Publishing House.

[4] Do Dinh Hoan (Editor), Nguyen Ang, Do Tien Dat, Phan Thanh Tam (2020), Math Exercise 5, Vietnam Education Publishing House.

[5] Tran Phuong, Nguyen Duc Tan, Pham Xuan Tien (2011), Primary Selected Mathematics (Volume 2: Math problems), Vietnam Education Publishing House.