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LAGRANGE DUALITY FOR GENERAL CONSTRAINED EXTREMUM PROBLEM

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Abstract:

The general constrained extremum problem is studied, in this paper, for which the given cone with its interior being empty. Making use of the Lagrange duality theory with a class of regular weak separation functions in the image space, i.e., the space where the images of the objective and constraint functions run, a sufficient optimality condition for a global minimum point of that problem is presented. In addition, we give an equivalent condition for a class of regular weak separation functions. The result obtained in the literature is new and also illustrated by an example for our findings.

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ĐỐI NGẪU LANGRANGE CHO BÀI TOÁN CỰC TRỊ CÓ RÀNG BUỘC TỔNG QUÁT

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Từ khóa:

Bài toán cực trị có ràng buộc tổng quát, Điểm cực tiểu toàn cục, Điều kiện tối ưu đủ, Đối ngẫu Langrange, Tập ảnh.

Tóm tắt

Trong bài báo, bài toán cực trị có ràng buộc tổng quát được nghiên cứu trong đó một nón cho trước có phần trong bằng rỗng. Vận dụng lý thuyết đối ngẫu Langrange cho một lớp các hàm tách được yếu chính quy trong không gian ảnh (tương ứng không gian ảnh mà ở đó các hàm mục tiêu và ràng buộc chuyển động), một điều kiện tối ưu đủ cho một điểm cực tiểu toàn cục của bài toán liên quan được cung cấp. Bên cạnh, chúng tôi cung cấp một điều kiện tương đương cho một lớp của các hàm tách yếu chính quy. Kết quả đạt được trong bài báo là mới và được mô tả bằng một ví dụ cụ thể nhằm mô tả các kết quả tìm được.

1. Introduction

Some notations of regular weak seperation mappings for real-valued functions and several basic calculus rules of which were introduced in the literature, see, e.g., Giannessi [4], Moldovan and Pellegrini [5, 6], Borrwein and Lewis [7], Rockarfellar [2,8], Clarke [3], Tan and Minh [1] and the cited REFERENCES therein. There are a lot of papers on all aspects of primal sufficient optimality conditions for global minimum points for a class of general constrained extremum problems in terms of regular weak seperation mappings in most of the aforementioned literatures.

In recent years, Lagrange duality theory plays an important role in the theory of optimization and espescially, it plays a cructial role in the theory of extremum problems, see [1,2,3,4,5,6,7,8] and the cited REFERENCES therein.

Note that the general Lagrange duality theory can be drawn from a seperation scheme. Furthermore, the class of regular weak seperation mappings is the natural ground for obtaining sufficient efficiency for global minimum points. Our main aim in this paper is to provide a sufficient optimality condition for a global minimum point for the general constrained extremum problem and then, we also introduce an equivalent condition for the class of regular weak seperation mappings. An example is also proposed to illustrate the main result of the paper.

2. Lagrangian duality

2.1. Preliminaries

Let us call R^n (where $R^n = \left\{ x = (x_1, ..., x_n) : x_i \in R,\ i = \overline{1,n} \right\}$) be the n-dimensional Euclidean space equipped with the usual Euclidean norm $\| . \|$ and further for any natural number n, we denote by I:={1,2,…, n}. The non-negative orthant cone of R^n is given by the following set

$$R_+^n = \left\{ x = (x_1,...,x_n) \in R^n : x_i \geq 0,\ i = \overline{1,n} \right\}.$$

The topological interior of the non-negative orthant cone of R^n is expressed as

$$R_{++}^n = \operatorname{int} R_+^n = \left\{ x = (x_1,...,x_n) \in R^n : x_i > 0,\ i = \overline{1,n} \right\}.$$

For each C $\subseteq$X, we denote as usual the symbols intC and clC instead of the interior and the closure C, respectively. Given a real-valued function $\varphi : R \times R^m \to R.$

The positive and non-positive level sets are defined respectively by

$$lev_{>0}\varphi = \left\{ (u,v) \in R \times R^m : \varphi(u,v) > 0 \right\}$$

and $lev_{\leq 0}\varphi = \left\{ (u,v) \in R \times R^m : \varphi(u,v) \leq 0 \right\}.$

The negative and non-negative level sets are defined respectively by

$$lev_{<0}\varphi = \left\{ (u,v) \in R \times R^m : \varphi(u,v) < 0 \right\}$$

and $lev_{\geq 0}\varphi = \left\{ (u,v) \in R \times R^m : \varphi(u,v) \geq 0 \right\}.$

For the investigation, let us consider the mappings

$$f : C \to R,\ g : C \to R^m,\ h : C \to R^p$$

Until now, we always accept a cone $D = R_+^m \times O_p$, a zero element $O_p = (0,0,...,0) \in R^p$, and then, we define the vector-valued mapping $\hat{g} : C \to R^{m+p},\ \hat{g}(x) = (g(x), h(x)),\ x \in C.$

Let us now consider a general constrained extremum problem in the following format:

$$f^{\downarrow} := \min f(x),$$
$$x \in K := \left\{ x \in C : g(x) \in R_+^m,\ h(x) = 0 \right\}. \quad (2.1)$$

Then the feasible region for the problem (2.1) has form: $K := \left\{ x \in C : \hat{g}(x) \in D \right\}.$

The polar cone of D is D^+, is given by $D^+ := \left\{ \delta \in R^{m+p} : \langle \delta, d \rangle \geq 0\ \ \forall d \in D \right\}.$

The dual problem of problem (2.1) is defined by the following format:

$$\sup_{\lambda \in D^+} \inf_{x \in C} \left[f(x) - \langle \lambda, \hat{g}(x) \rangle \right]. \quad (2.2)$$

By symmetry, the problem (2.1) is associated with the following problem

$$\inf_{x \in C} \sup_{\lambda \in D^+} \left[f(x) - \langle \lambda, \hat{g}(x) \rangle \right]. \quad (2.3)$$

Obviously, we always have

$$\sup_{\lambda \in D^+} \inf_{x \in C} \left[f(x) - \langle \lambda, \hat{g}(x) \rangle \right] \leq \inf_{x \in C} \sup_{\lambda \in D^+} \left[f(x) - \langle \lambda, \hat{g}(x) \rangle \right]. \quad (2.4)$$

The difference among the right-hand and left-hand sides of (2.4) is said to be duality gap. The dual problem for the problem (2.2) may be reached by putting condition (2.2) in the format (2.3), namely:

$$-\inf_{\lambda \in D^+} \sup_{x \in C} \left[-f(x) + \langle \lambda, \hat{g}(x) \rangle \right].$$

In this case, the dual problem of (2.2) coincides with the problem (2.3) because:

$$-\sup_{x \in C} \inf_{\lambda \in D^+} \left[-f(x) + \langle \lambda, \hat{g}(x) \rangle \right] = \inf_{x \in C} \sup_{\lambda \in D^+} \left[f(x) - \langle \lambda, \hat{g}(x) \rangle \right].$$

Definition 2.1: Let us arbitrarily consider $\bar{x} \in K$. The set

$$K_{\bar{x}} := \left\{ (u,v) \in R \times R^m : u = f(\bar{x}) - f(x),\ v = \hat{g}(x),\ x \in C \right\}$$

is called the image of C through the mapping $A_{\bar{x}} : C \to R^m \times R$, given by

$$A_{\bar{x}}(x) := \left(f(\bar{x}) - f(x),\ \hat{g}(x) \right),\ x \in C.$$

Definition 2.2 [3]: Xét một tập các tham số Γ và lớp các hàm $w : R \times R^m \times \Gamma \to R$, cho bởi

$$\mathrm{w}(u,v;\gamma) = u + \underline{w}(v;\gamma),\quad \gamma \in \Gamma, \quad (2.5)$$

ở đây $w : R^m \times \Gamma \to R$, cho bởi

$$\forall v \in R^m \ \forall \gamma \in \Gamma \ \forall \alpha \geq 0 \ \exists \gamma_\alpha \in \Gamma: \quad \alpha \underline{w}(v;\gamma) = \underline{w}(v;\gamma_\alpha). \quad (2.6)$$

The functions of (2.5) is called the regular weak separation functions, iff:

$$\bigcap_{\gamma \in \Gamma} lev_{>0} w(.,.;\gamma) = R_{++} \times D.$$

2.2. New results of the paper

Our main purpose of this section is to provide a sufficient optimality condition for a global minimum point of problem (2.1) based on a separation scheme.

A sufficent optimality condition under a suitable assumption on the regular weak separation functions is presented in the following theorem:

Theorem 3.1: (Sufficent optimality condition for a global minimum**)** Let us consider a point $\bar{x} \in K$. If the following equality holds

$$\max_{\lambda \in D^+} \inf_{x \in C} \left[f(x) - \langle \lambda, g(x) \rangle \right] = f(\bar{x}), \quad (3.1)$$

then $\bar{x} \in K$ is a global minimum for the extremum problem (2.1).

Proof: We define the real-valued function $w: R \times R^m \times D^+ \to R,$ given by

$$w(u,v;\gamma) = u + \underline{w}(v;\gamma), \quad \gamma \in D^+, \quad (3.2)$$

where $\underline{w}: R^m \times D^+ \to R,$ defined by $\underline{w}(v;\gamma) = \langle \gamma, v \rangle, \ \gamma \in D^+$. It is evident that

$$\forall v \in R^m \ \forall \gamma \in D^+ \ \forall \alpha \geq 0 \ \exists \gamma_\alpha = \alpha\gamma \in D^+: \ \alpha \underline{w}(v;\gamma) = \underline{w}(v;\gamma_\alpha).$$

(3.2*)

Under the hypothese (2.6), the elements as constructed above is regular weak seperation because

$$\bigcap_{\gamma \in D^+} lev_{>0} w(.,.;\gamma) = R_{++} \times D.$$

It is not difficult to see that condition (3.1) is equivalent to

$$\max_{\lambda \in D^+} \inf_{x \in C} \left[f(x) - f(\bar{x}) - \langle \lambda, g(x) \rangle \right] = 0,$$

or equivalently, if:

$$\min_{\gamma \in D^+} \sup_{(u,v) \in K_{\bar{x}}} \left[u + \underline{w}(v;\gamma) \right] = 0. \quad (3.3)$$

Therefore, the exists $\bar{\gamma} \in D^+$ such that $\sup_{(u,v) \in K_{\bar{x}}} \left[u + \underline{w}(v;\bar{\gamma}) \right] = 0$. By viture of the notion of supremum, one can see achieve that $u + \underline{w}(v;\bar{\gamma}) \leq 0, \ \forall (u,v) \in K_{\bar{x}},$ means that there exists $\bar{\gamma} \in D^+$ such that

$$u + \underline{w}(v;\bar{\gamma}) \leq 0, \quad \forall (u,v) \in K_{\bar{x}}.$$

(3.4)

Thanks to the concept of the class of regular weak seperation funtions, we deduce that

$$\bigcap_{\gamma \in D^+} lev_{>0} w(.,.;\gamma) = R_{++} \times D,$$

which ensures that

$$R_{++} \times D \subseteq lev_{>0} w(.,.;\bar{\gamma}). \quad (3.5)$$

Taking arbitrary $(u,v) \in K_{\bar{x}},$ which implies that $w(u,v;\gamma) = u + \underline{w}(v;\gamma) \leq 0,$ means that

$$(u,v) \in lev_{\leq 0} w(.,.;\bar{\gamma}).$$

Thus,

$$K_{\bar{x}} \subseteq lev_{\leq 0} w(.,.;\bar{\gamma}). \quad (3.6)$$

Now noting that

$$lev_{>0} w(.,.;\bar{\gamma}) \cap lev_{\leq 0} w(.,.;\bar{\gamma}) = \phi. \quad (3.7)$$

Combining (3.5), (3.6) and (3.7) yields that

$$K_{\bar{x}} \cap R_{++} \times D = \phi,$$

and this leads to the impossibility of the system:

$$f(\bar{x}) - f(x) > 0, \ \ g(x) \geq 0, \ \ h(x) = 0, \ x \in C.$$

Consequently, the point $\bar{x} \in K$ is a global minimum point for the extremum problem (2.1) and we arrive at the desired conclusion.

Theorem 3.1 is illustrate by the following example:

Example 3.3: Let us consider the extremum problem (2.1) for which $C = [1, +\infty) \times R_+,$ $D = R_+^2$, the mapping

$$f: C \to R,\ f\left(\left(x,y\right)\right) = x^3 + y^3 - 3x^2 + 6x + 2y - 3$$

for every $(x,y) \in C$, the mapping

$$g: C \to R^2,\ g\left(\left(x,y\right)\right) = \left(x^3 - 1,\ y^3 + y - x + 1\right)$$

for every $(x,y) \in C$.

Then the feasible region of the extremum problem (2.1) is *K=C*, where

$$K = \left\{(x,y) \in C : g\left((x,y)\right) \geq 0\right\}.$$

Directly calculating gives that

$$D^+ = \left\{(\lambda,\mu) \in R^2 : \lambda x + \mu y \geq 0\ \forall (x,y) \in R_+^2\right\} = R_+^2.$$

For the investigation, let us consider a point $\bar{x} = (1,0) \in K$, it holds that $f(\bar{x}) = 1$. In the sequel, for any element $(x,y) \in C$ and for any Lagrange multiplier $(\lambda,\mu) \in D^+$, one can obtain the following result

$$\begin{aligned} & f(x,y) - \left\langle \lambda, g(x,y)\right\rangle \\ & = (x-1)^3 + 3(x-1) + y^3 + 2y + 1 \\ & \quad + \lambda\left(x^3 - 1\right) + \mu\left(y^3 + y - (x-1)\right) \\ & \geq 1 + \mu(1-x). \end{aligned}$$

Therefore,

$$\inf_{(x,y)\in C}\left(f(x,y) - \left\langle \lambda, g(x,y)\right\rangle\right) = 1 + \mu(1-x) \leq 1,$$

and the equality holds at x=1, y =0. Consequently,

$$\max_{(\lambda,\mu)\in D^+}\inf_{(x,y)\in C}\left(f(x,y) - \left\langle(\lambda,\mu), g(x,y)\right\rangle\right) = \max_{(\lambda,\mu)\in D^+}\left(1 + \mu(1-x)\right) = 1$$

achieved at $\mu = 0,\ \lambda \geq 0$ arbitrarily. Thus, the condition (3.1) in Theorem 2.1 is fulfilled. From her, we assert that $\bar{x} = (1,0)$ is a global minimum point for the extremum problem (2.1).

In fact, in this setting, in view of the definition one has for any feasible poin $(x,y) \in K$,

$$\begin{aligned} f\left((x,y)\right) &= x^3 + y^3 - 3x^2 + 6x + 2y - 3 \\ &= (x-1)^3 + 3(x-1) + y^3 + 2y + 1 \geq 1 \\ &= f(\bar{x}). \end{aligned}$$

They mean that $\bar{x} = (1,0)$ a global minimum point for the extremum problem (2.1), as required.

To the end of this section, we derive a necessary condition for the class of regular weak seperation funtions.

Theorem 3.2 (An equivalent condition for the class of regular weak seperation funtions): Let us consider a parameters set Γ and a class of elements $w: R \times R^m \times \Gamma \to R$, given by (2.5) in which the condition (2.6) is valid. Then, the elements of (2.5) are regular weak seperation funtions if and only if

$$\bigcap_{\gamma\in\Gamma} lev_{\geq 0}\underline{\mathrm{w}}(.;\gamma) = D.$$

Proof: "$\Rightarrow$":

We set $A = \bigcap_{\gamma\in\Gamma} lev_{>0}\mathrm{w}(.,.;\gamma)$ and for each $v \in R^m$, $A_v = \bigcap_{\gamma\in\Gamma} lev_{\geq 0}\underline{\mathrm{w}}(.;\gamma)$.

We need to show that (*) $R_{++} \times D \subseteq A.$ Indeed, for every $(u,v) \in R_{++} \times D$ it results in

$$u + \underline{w}(v;\gamma) > 0,\quad \forall \gamma \in \Gamma.$$

For all $\gamma \in \Gamma$, it holds that

$$\begin{aligned} & lev_{>0}\mathrm{w}(.,.;\gamma) \\ & = \left\{(u,v) \in R \times R^m : \mathrm{w}(u,v;\gamma) = u + \underline{w}(v;\gamma) > 0\right\} \end{aligned}$$

and hence $(u,v) \in A.$ We mean that condition (*) is satisfied. Now, we suppose to the contrary, that there exists $\left(\hat{u},\hat{v}\right) \in A \setminus \left(R_{++} \times D\right)$ satisfying

(**) $\mathrm{w}\left(\hat{u},\hat{v};\gamma\right) = \hat{u} + \underline{w}\left(\hat{v};\gamma\right) > 0$ for all $\gamma \in \Gamma$.

In other words, by definition there exists $\gamma_0 \in \Gamma$ such that $\underline{w}\left(\hat{v};\gamma_0\right) = 0.$ We consider two cases can occur as follows:

Case 1: If $\hat{v} \in D$, it follows from the hypotheses that $\left(\hat{u},\hat{v}\right) \notin \left(R_{++} \times D\right)$ and hence $\hat{u} \leq 0$. In addition, $\underline{w}\left(\hat{v};\gamma\right) > -\hat{u} \geq 0$ for any $\gamma \in \Gamma$, this is a contradiciton!.

Case 2: If $\hat{v} \notin D$, it yields from the assumption $A_v = D$ that the exists $\hat{\gamma} \in \Gamma$ such that $\underline{w}(\hat{v};\hat{\gamma}) < 0.$ Note that in the case when $\hat{u} \le 0$ then $w(\hat{u},\hat{v};\hat{\gamma}) = \hat{u} + \underline{w}(\hat{v};\hat{\gamma}) < 0$, this inequality conflicts with the inequality (**). In the case when $\hat{u} > 0$, according to the condition (2.6) where

$$v = \hat{v},\ \gamma = \hat{\gamma},\ \alpha = \bar{\alpha} = -\frac{\hat{u}}{\underline{w}(\hat{v};\hat{\gamma})} > 0,$$

which leads to the existence of $\gamma_{\bar{\alpha}} \in \Gamma$ such that the condition (3.2*) is valid. This is a contradiction with (**) because $\hat{u} + \underline{w}(\hat{v};\gamma_{\bar{\alpha}}) = \hat{u} + \bar{\alpha}\underline{w}(\hat{v};\hat{\gamma}) = 0.$

$"\Leftarrow"$:

Suppose to the contrary, that $A_v \ne D$ which follows that, either $A_v \setminus D \ne \phi$ or $D \setminus A_v \ne \phi.$ For the first case, there would exists an element $\hat{v} \in A_v \setminus D,\ \hat{v} \in A_v$ which implies that $\underline{w}(\hat{v};\hat{\gamma}) \ge 0\ \ \forall \hat{\gamma} \in \Gamma.$ Taking arbitrary $u > 0$, one can be achieve the following result

$$w(u,\hat{v};\hat{\gamma}) = u + \underline{w}(\hat{v};\hat{\gamma}) > 0 \quad \forall \hat{\gamma} \in \Gamma,$$

we mean that

$$(u,\hat{v}) \in \bigcap_{\gamma \in \Gamma} lev_{>0} w(.,.;\gamma) = A,\ \ A = R_{++} \times D.$$

This is a contradiction. For the later case, there exists $\hat{v} \in D \setminus A_v,\ \hat{v} \notin A_v$ which yields that

$$\exists \hat{\gamma} \in \Gamma :\ \underline{w}(\hat{v};\hat{\gamma}) < 0.$$

Taking *u* such that $u \in \left(0, -\underline{w}(\hat{v};\hat{\gamma})\right),$ we obtain that $(u,\hat{v}) \in A$ and furthermore

$$w(u,\hat{v};\hat{\gamma}) = u + \underline{w}(\hat{v};\hat{\gamma}) < 0,\ \ \hat{\gamma} \in \Gamma,$$

we arrive at a contradiction!

The propositon is proved completely.

3. Conclusion

Based on the concept of the elements of the format (2.5) are regular weak seperation funtions and the notion of image sets, we have obtained a sufficient optimality condition for a global minimum point for the extremum problem (2.1) via the Lagrangian dual model. Besides, we also gave a necessary and sufficient condition for the class of regular weak seperation funtions.

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