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A NOVEL PROJECTION TECHNIQUE FOR SOLVING PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Abstract:

The general constrained extremum problem is studied, in this paper, for which the given cone with its interior being empty. Making use of the Lagrange duality theory with a class of regular weak separation functions in the image space, i.e., the space where the images of the objective and constraint functions run, a sufficient optimality condition for a global minimum point of that problem is presented. In addition, we give an equivalent condition for a class of regular weak separation functions. The result obtained in the literature is new and also illustrated by an example for our findings.

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MỘT KỸ THUẬT MỚI CỦA PHÉP CHIẾU GIẢI BÀI TOÁN CÂN BẰNG GIẢ ĐƠN ĐIỆU

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Tóm tắt

Trong bài báo này chúng tôi phân tích một phương pháp mới giải bài toán cân bằng với song hàm giả đơn điệu và thỏa mãn điều kiện kiểu Lipschitz. Định lý hội tụ mạnh được trình bày không cần biết trước thông tin hằng số Lipschitz của song hàm. Cuối cùng, một vài ví dụ số được đưa ra để minh họa hiệu suất của thuật toán đề nghị.

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and $f: H \times H \to \mathbb{R}$ be a bifunction with $f(x,x) = 0$ for all $x \in C$. The equilibrium problem (**EP**) for the bifunction f on C is stated as follows:

Find $x^* \in C$ such that $f(x^*, y) \geq 0$ for all $y \in C$. (1)

Let us denote $EP(f,C)$ by the solution set of the problem (**EP**). To the best of our knowledge, the term "equilibrium problem" introduced in 1992 by Muu and Oettli [25] and has been further studied by Blum and Oettli [3]. Equilibrium problem is also called the Ky Fan inequality due to his contribution to this field [8].

In fact, the problem (**EP**) is a generalization of many mathematical models including variational inequality problems, optimization problems and fixed point problems, see, [3, 16, 17, 25]. The problem (**EP**) has been considered by many authors in recent years, see, [12,14,10,18,21,22, 24,26,27,30,31,33] and the REFERENCES therein.

By using the idea of Korpelevich extragradient method [15], Flam et al. [9] and Quoc et al. [30] proposed the following algorithm for solving equilibrium problem involving pseudomonotone and Lipschitz-type bifunction:

$$\begin{cases} y_n = \arg\min\left\{\lambda f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\right\} \\ x_{n+1} = \arg\min\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\right\} \end{cases} \quad \textbf{(EGM)}$$

where $\lambda > 0$ is a suitable parameter. The methods in [9,30] are also called the extra-gradient method (**EGM**) due to the result of Korpelevich in [15]. In recent years, the extra-gradient method has improved and extended by many authors see, e.g., [13, 20, 27, 35, 36].

Observe that some known methods use the constant stepsizes which depend on the Lipschitztype constants of the bifunctions [23,34,35]. This fact can give some restrictions in applications because the Lipschitz-type constants are often unknown or difficult to estimate.

In [11], Hieu et al. recently proposed two algorithms [11, Algorithm 3.1, Algorithm 4.1] for solving an equilibrium problem whose associated bifunction is monotone and satisfies a Lipschitztype condition in a Hilbert space. In the first algorithm in [11], it is assumed that the value of the Lipschitz constant of the bifunction is known while in the second one the prior knowledge of this constant or its estimate is not explicitly needed. The proposed algorithms are constructed around the proximal-like mapping and the regularized method and use some new variable step size rules. Strong convergence theorems are established under some mild conditions imposed on bifunction and control parameters. Finally several numerical results are provided to illustrate the behavior of the new algorithms and to compare them to well-known algorithms.

We comment here that the methods of [11] require computing the proximal-like mapping twice per iteration and this could be costly especially since one needs to solve an optimization problem twice per iteration during implementation. This defect also occurred in [35, Algorithm 1]. Furthermore, the bifunction is compelled to be monotone in [11, Algorithm 3.1, Algorithm 4.1] which excludes some other important class of the bifunctions (pseudo-monotone bifunctions). These are setbacks in the methods proposed in [11].

In recent years, inertial type algorithms can be regarded as a technique to speed up the convergence properties have received a lot of attention from many authors for solving optimization problems, variational inequality problems and monotone inclusions, see, [1,2,7,19,24,29] and the

REFERENCES therein. So, a natural question which raises is:

Is it possible to introduce a new strongly convergent extra-gradient algorithm with inertial effects for finding a solution of EP (1) with pseudo-monotone bifunction which does not depend on the Lipschitz-type constants of the bifunctions?

In this work, we give a positive answer to this question. Motivated and inspired by the works of Censor et al. [4] and Lyashko et al. [19], we will propose a new extra-gradient type algorithm for finding a solution of the EP in the setting of infinite-dimensional real Hilbert spaces.

This paper is organized as follows: In Sect. 2, we collect some definitions and preliminary results for further use and then propose a new algorithm in the more details. Sect. 3 deals with analyzing the convergence of the proposed algorithm. Sect. 4 gives several numerical results on two test problems to illustrate the convergence of the algorithm and compare it with studied algorithms.

2 Preliminaries

Let C be a nonempty closed convex subset of H. We begin with some concepts of monotonicity of a bifunction [3,25].

Definition 1.1. A bifunction $f: H \times H \to \mathbb{R}$ is said to be:
(1) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x,y) + f(y,x) \leq -\gamma \parallel x - y \parallel^2,\ \forall x,y \in C;$$

(2) monotone on C if $f(x,y) + f(y,x) \leq 0$ for all $x,y \in C$;

(3) pseudomonotone on C if $f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0$ for all $x,y \in C$;

Definition 1.2. A bifunction $f: H \times H \to \mathbb{R}$ is said to satisfy the Lipschitz-type condition on C if there exist two positive constants c_1, c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 \parallel x - y \parallel^2 - c_2 \parallel y - z \parallel^2,\ \forall x,y,z \in C.$$

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{w \in H: \langle w, x - y\rangle \geq 0, \forall y \in C\}.$$

For all $x \in H$, the metric projection $P_C x$ of x onto C is defined by

$$P_C x = \arg\min\{\parallel y - x \parallel : y \in C\}.$$

Since C is nonempty closed and convex, $P_C x$ exists and is unique.

Lemma 2.1. [28, Proposition 3.61] *Let C be a nonempty closed convex subset of H and $g: H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function on H. Assume either that g is continuous at some point of C, or that there is an interior point of C where g is finite. Then, x^* is a solution to the following convex problem $min\{g(x): x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 2.2. ([32]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,\ \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying*

$$\liminf_{k\to\infty} f\left(a_{n_k+1} - a_{n_k}\right) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. ([5]) *Let H be a real Hilbert space. Then the following result holds:*

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, x + y\rangle \ \forall x, y \in H.$$

3 Main results Now, we introduce a new algorithm for solving the problem (**EP**).

Algorithm 3.1.

Initialization: *Let $\theta > 0, \tau_1 > 0, \mu \in (0,1)$ and $x_0, x_1 \in H$ be arbitrary. Let $\{\lambda_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \lambda_n < +\infty$.*

We assume that $\{\theta_n\}$ and $\{\epsilon_n\}$ are two positive sequences such that $\{\theta_n\} \subset [0, \theta)$ and $\epsilon_n = o(\alpha_n)$, means $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and $x_n (n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \mathrm{m}\ \left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \quad (2)$$

Step 2. Given the current iterates x_{n-1} and x_n for each $n \geq 1$, compute

$$\begin{cases} w_n = (1 - \alpha_n)\left(x_n + \theta_n(x_n - x_{n-1})\right), \\ y_n = \operatorname*{argmin}_{y\in C} \left\{\tau_n f(w_n, y) + \dfrac{1}{2}\|y - w_n\|^2\right\}. \end{cases}$$

If $y_n = w_n$, then stop and y_n is a solution. Otherwise, go to Step 3 .

Step 3. Select $v_n \in \partial f(w_n,\cdot)(y_n)$ and $q_n \in N_C(y_n)$ satisfying

$$q_n = w_n - \tau_n v_n - y_n \quad (3)$$

and construct a half-space

$$T_n = \{z \in H: \langle w_n - \tau_n v_n - y_n, z - y_n\rangle \leq 0\},$$

compute

$$x_{n+1} = \operatorname*{argmin}_{y\in T_n} \left\{\tau_n f(y_n, y) + \frac{1}{2}\|y - w_n\|^2\right\}$$

And

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\mu}{2} \dfrac{\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2}{f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1})}, \tau_n + \lambda_n\right\} \\ \qquad\qquad if\ f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1}) > 0, \\ \tau_n + \lambda_n \qquad \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return to setp 1.

In order to establish the strong convergence of Algorithm 3.1, we assume that the bifunction $f: H \times H \to R$ satisfies the following conditions:

Condition 3.1.

(A1) f is pseudomonotone on C;

(A2) f satisfies the Lipschitz-type condition on H with two constants c_1 and c_2;

(A3) $f(\cdot, y)$ is sequentially weakly upper semi-continuous on C for each fixed point $y \in C$, i.e.,

if $\{x_n\} \subset C$ is a sequence converging weakly to $x \in C$, then

$$\limsup_{n\to\infty} f(x_n, y) \leq f(x, y)$$

(A4) $f(x,\cdot)$ is convex, lower semi-continuous on H for every fixed $x \in H$.

(A5) Either $intC \neq \emptyset$ or $f(x, .)$ is continuous at some point in C for every $x \in H$.

Remark 3.1. From the conditions (A1) and (A2), we get $f(x, x) = 0$ for all $x \in C$.

It is easy to show that, under Condition 1, the solution set $EP(f, C)$ of the problem **(EP)** is closed and convex (see, for instance, [30]).

In this section, we analyze the convergence of Algorithm 3.1. We start with the following Remark and Lemmas which play an important role in proving the convergence of the proposed algorithm.

Remark 3.2.

1. Thanks to Lemma 2.1, there always exists $v_n \in \partial f(w_n, \cdot)(y_n)$ and $q_n \in N_C(y_n)$ such that (3) is satisfied. Hence, Algorithm 3.1 is well-defined.

2. With the selection in **Step 3** it is easy to show that $C \subset T_n$.

Lemma 3.1. ([37]) *Let $\{\tau_n\}$ be a sequence generated by Algorithm 3.1. Then $lim_{n\to\infty}\tau_n = \tau \in$*

$$\left[min\left\{\frac{\mu}{2\,\text{m}\ \{c_1, c_2\}}, \tau_1\right\}, \tau_1 + \lambda\right], \textit{where } \lambda = \sum_{n=1}^{\infty} \lambda_n. \textit{ Moreover, we obtain}$$

$$f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu}{2\tau_{n+1}}\left(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\right). \quad (4)$$

Let us claim the main result of this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of H and $f: H \times H \to \mathbb{R}$ be a bifunction satisfying Condition 3.1. In addition, we assume that the solution set $EP(f, C)$ is nonempty. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to an element $u \in EP(f, C)$, where $\| u \| = min\{\| z \|: z \in EP(f, C)\}$.*

Proof: **Claim 1.** The sequence $\{x_n\}$ is bounded. Indeed, first we show that, there exists $n_0 \in \mathbb{N}$ such that

$$\|x_{n+1} - u\| \leq \|w_n - u\| \ \ \forall n \geq n_0.$$

We have, since

$$x_{n+1} = \underset{y\in T_n}{\operatorname{argmin}}\left\{\tau_n f(y_n, y) + \frac{1}{2}\|y - w_n\|^2\right\}$$

and Lemma 2.1, it follows that

$$\tau_n\big(f(y_n, y) - f(y_n, x_{n+1})\big) \geq \langle w_n - x_{n+1}, y - x_{n+1}\rangle, \ \forall y \in T_n. \quad (5)$$

Let $u \in EP(f, C) \subset C \subset T_n$ and $y := u$, we obtain

$$\tau_n\big(f(y_n, u) - f(y_n, x_{n+1})\big) \geq \langle w_n - x_{n+1}, u - x_{n+1}\rangle. \quad (6)$$

Since $u \in EP(f, C) \subset C$ and $y_n \in C$, we have $f(u, y_n) \geq 0$. By the pseudomonotonicity of f, we obtain $f(y_n, u) \leq 0$, which implies from (6) that

$$-\tau_n f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, u - x_{n+1}\rangle. \quad (7)$$

Note that, from $v_n \in \partial f(w_n, \cdot)(y_n)$, we get

$$f(w_n, y) - f(w_n, y_n) \geq \langle v_n, y - y_n\rangle, \ \forall y \in H.$$

In particular, substituting $y := x_{n+1}$, we get

$$\tau_n\big(f(w_n, x_{n+1}) - f(w_n, y_n)\big) \geq \tau_n\langle v_n, x_{n+1} - y_n\rangle. \quad (8)$$

By the definition of T_n, we have

$$\langle w_n - \tau_n v_n - y_n, x_{n+1} - y_n\rangle \leq 0$$

and so

$$\tau_n\langle v_n, x_{n+1} - y_n\rangle \geq \langle w_n - y_n, x_{n+1} - y_n\rangle. \quad (9)$$

Combining (8) and (9), we obtain

$$\tau_n\big(f(w_n, x_{n+1}) - f(w_n, y_n)\big) \geq \langle w_n - y_n, x_{n+1} - y_n\rangle. \quad (10)$$

Adding (7) and (10), we get

$$\begin{aligned}
2\tau_n\big(f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1})\big) \geq\ & 2\langle w_n - y_n, x_{n+1} - y_n\rangle + 2\langle w_n - x_{n+1}, u - x_{n+1}\rangle \\
=\ & \big(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_{n+1} - w_n\|^2\big) \\
& +\big(\|w_n - x_{n+1}\|^2 + \|x_{n+1} - u\|^2 - \|w_n - u\|^2\big) \\
=\ & \|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 + \|x_{n+1} - u\|^2 - \|w_n - u\|^2.
\end{aligned}$$

This implies that

$$\|x_{n+1} - u\|^2 \leq \|w_n - u\|^2 - \|w_n - y_n\|^2 - \|x_{n+1} - y_n\|^2 + 2\tau_n\big(f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1})\big). \quad (11)$$

From (4), we get

$$2\big(f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1})\big) \leq \frac{\mu}{\tau_{n+1}}\big(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\big). \quad (12)$$

Substituting (11) into (12), we get

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \|w_n-u\|^2-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n-y_n\|^2-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|x_{n+1}-y_n\|^2\\ &= \|w_n-u\|^2-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\left(\|w_n-y_n\|^2+\|x_{n+1}-y_n\|^2\right). \quad (13)\end{aligned}$$

We also have $\lim_{n\to\infty}\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)=1-\mu>\frac{1-\mu}{2}$, thus there exists $n_0\in\mathbb{N}$ such that $1-\mu\frac{\tau_n}{\tau_{n+1}}>0\ \forall n\ge n_0$, by (13) we obtain

$$\|x_{n+1}-u\|\le\|w_n-u\|\ \ \forall n\ge n_0.$$

On the other hand, we have

$$\begin{aligned}\|w_n-u\| &= \|(1-\alpha_n)\big(x_n+\theta_n(x_n-x_{n-1})\big)-u\|\\ &= \|(1-\alpha_n)(x_n-u)+(1-\alpha_n)\theta_n(x_n-x_{n-1})-\alpha_n u\|\\ &\le (1-\alpha_n)\|x_n-u\|+(1-\alpha_n)\theta_n\|x_n-x_{n-1}\|+\alpha_n\| u \|\\ &= (1-\alpha_n)\|x_n-u\|+\alpha_n\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\| u \|\right]. \quad (14)\end{aligned}$$

Moreover, since (2) we have

$$\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\le\frac{\epsilon_n}{\alpha_n}\to 0,$$

this implies that $\lim_{n\to\infty}\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\| u \|\right]=\| u \|$, thus there exists $M>0$ such that

$$(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\| u \|\le M \quad (15)$$

Combining (14) and (15) we obtain

$$\|w_n-u\|\le(1-\alpha_n)\|x_n-u\|+\alpha_n M.$$

Thus

$$\begin{aligned}\|x_{n+1}-u\|\le &\ (1-\alpha_n)\|x_n-u\|+\alpha_n M\\ &= \max\{\|x_n-u\|,M\}\le\cdots\le\max\{\|x_{n_0}-u\|,M\}.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded.

Claim 2.

$$\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n-w_n\|^2+\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|x_{n+1}-y_n\|^2\le\|x_n-u\|^2-\|x_{n+1}-u\|^2+\alpha_n M_1.$$

Indeed, we have $\|w_n-u\|\le(1-\alpha_n)\|x_n-u\|+\alpha_n M$, this implies that

$$\begin{aligned}\|w_n-u\|^2 &\le (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n(1-\alpha_n)M\|x_n-u\|+\alpha_n^2M^2\\ &\le \|x_n-u\|^2+\alpha_n[2(1-\alpha_n)M\|x_n-u\|+\alpha_n M^2]\\ &\le \|x_n-u\|^2+\alpha_n M_1, \quad (16)\end{aligned}$$

where $M_1:=\max\{2(1-\alpha_n)M\|x_n-u\|+\alpha_n M^2\colon n\in\mathbb{N}\}$. Substituting (16) into (13) we get

$$\|x_{n+1}-u\|^2\le\|x_n-u\|^2+\alpha_n M_1-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n-w_n\|^2-\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|x_{n+1}-y_n\|^2,$$

or equivalently

$$\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n-w_n\|^2+\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|x_{n+1}-y_n\|^2\le\|x_n-u\|^2-\|x_{n+1}-u\|^2+\alpha_n M_1.$$

Claim 3.

$$\begin{aligned}\|x_{n+1}-u\|^2\le &\ (1-\alpha_n)\|x_n-u\|^2+\alpha_n\Big[2(1-\alpha_n)\|x_n-u\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\ &+\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\| z \|\,\|w_n-x_{n+1}\|+2\langle -u,x_{n+1}-u\rangle\Big].\end{aligned}$$

Indeed, using Lemma 2.3 we get

$$\begin{aligned}
\|x_{n+1}-u\|^2 \leq\ & \|w_n-u\|^2 \\
=\ & \|(1-\alpha_n)(x_n-u)+(1-\alpha_n)\theta_n(x_n-x_{n-1})-\alpha_n u\|^2 \\
\leq\ & \|(1-\alpha_n)(x_n-u)+(1-\alpha_n)\theta_n(x_n-x_{n-1})\|^2+2\alpha_n\langle -u,w_n-u\rangle \\
=\ & (1-\alpha_n)^2\|x_n-u\|^2+2(1-\alpha_n)\theta_n\|x_n-u\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2 \\
& +2\langle -u,w_n-x_{n+1}\rangle+2\langle -u,x_{n+1}-u\rangle \\
\leq\ & (1-\alpha_n)\|x_n-u\|^2+\alpha_n\Big[2(1-\alpha_n)\|x_n-u\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| \\
& +\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\parallel u\parallel\|w_n-x_{n+1}\|+2\langle -u,x_{n+1}-u\rangle\Big]
\end{aligned}$$

Claim 4. $\{\|x_n-u\|^2\}$ converges to zero for each $n\geq 0$. Set

$$a_n:=\|x_n-u\|^2$$

and

$$b_n\coloneqq 2(1-\alpha_n)\|x_n-u\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|$$
$$+2\parallel u\parallel\|w_n-x_{n+1}\|+2\langle -u,x_{n+1}-u\rangle.$$

Then, **Claim 3** can be rewritten as follows:

$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n b_n$, satisfying

$$\liminf_{k\to\infty}\left(a_{n_k+1}-a_{n_k}\right)\geq 0.$$

This is equivalently to that we need to show

$$\limsup_{k\to\infty}\langle u,u-x_{n_k+1}\rangle\leq 0$$

and

$$\limsup_{k\to\infty}\|w_{n_k}-x_{n_k+1}\|\leq 0$$

for every subsequence $\{\|x_{n_k}-u\|\}$ of $\{\|x_n-u\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|\right)\geq 0.$$

Suppose that $\{\|x_{n_k}-u\|\}$ is a subsequence of $\{\|x_n-u\|\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|\right)\geq 0.$$

Then

$$\begin{aligned}
\liminf_{k\to\infty}\left(\|x_{n_k+1}-u\|^2-\|x_{n_k}-u\|^2\right) &= \liminf_{k\to\infty}\left[\left(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|\right)\left(\|x_{n_k+1}-u\|+\|x_{n_k}-u\|\right)\right] \\
&\geq 0.
\end{aligned}$$

By **Claim 2** we obtain

$$\begin{aligned}
\limsup_{k\to\infty} & \left[\left(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\right)\|w_{n_k}-y_{n_k}\|^2+\left(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\right)\|x_{n_k+1}-y_{n_k}\|^2\right] \\
& \leq \limsup_{k\to\infty}\left[\|x_{n_k}-u\|^2-\|x_{n_k+1}-u\|^2+\alpha_{n_k}M_1\right] \\
& \leq \limsup_{k\to\infty}\left[\|x_{n_k}-u\|^2-\|x_{n_k+1}-u\|^2\right]+\limsup_{k\to\infty}\alpha_{n_k}M_1 \\
& = -\liminf_{k\to\infty}\left[\|x_{n_k+1}-u\|^2-\|x_{n_k}-u\|^2\right] \\
& \leq 0
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\|x_{n_k+1}-y_{n_k}\|=0. \tag{17}$$

Since (17), we obtain

$$\lim_{k\to\infty}\|x_{n_k+1}-w_{n_k}\|=0. \tag{18}$$

Now, we show that

$$\|x_{n_k+1}-x_{n_k}\| \to 0 \text{ as } n \to \infty. \quad (19)$$

Indeed, we have

$$\|x_{n_k}-w_{n_k}\| = \theta_{n_k}\|x_{n_k}-x_{n_k-1}\| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\| \to 0. \quad (20)$$

Combining (18) and (20), we get

$$\|x_{n_k+1}-x_{n_k}\| \le \|x_{n_k+1}-w_{n_k}\| + \|w_{n_k}-x_{n_k}\| \to 0.$$

Since the sequence $\{x_{n_k}\}$ is bounded, without loss of generality we can assume that $\{x_{n_k}\}$ converges weakly to some $z^* \in H$, such that

$$\limsup_{k\to\infty}\langle -u, x_{n_k}-u\rangle = \langle -u, z^*-u\rangle. \quad (21)$$

Using (20), we get

$$w_{n_k} \to z^* \text{ as } k \to \infty.$$

Now, from $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\| = 0$, $\lim_{k\to\infty}\|w_{n_k}-x_{n_k+1}\| = 0$, $\lim_{k\to\infty}\|x_{n_k+1}-y_{n_k}\| = 0$, we will show that

$$z^* \in EP(f, C).$$

Indeed, from $w_{n_k} \to z^*$ and $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\| = 0$, we obtain $y_{n_k} \to z^*$ and so, since $\{y_n\} \subset C$, we have $z^* \in C$. Moreover, we have

$$2\big(f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1})\big) \le \frac{\mu}{\tau_{n+1}}\left(\|w_n-y_n\|^2 + \|x_{n+1}-y_n\|^2\right),$$

which follows that

$$2f(y_n, x_{n+1}) \ge 2\big(f(w_n, x_{n+1}) - f(w_n, y_n)\big) - \frac{\mu}{\tau_{n+1}}\left(\|w_n-y_n\|^2 + \|x_{n+1}-y_n\|^2\right). \quad (22)$$

Thus, from (10), it follows that

$$f(w_n, x_{n+1}) - f(w_n, y_n) \ge \frac{1}{\tau_n}\langle w_n - y_n, x_{n+1}-y_n\rangle. \quad (23)$$

Combining (22) and (23), we get

$$2f(y_n, x_{n+1}) \ge \frac{1}{\tau_n}\langle w_n - y_n, x_{n+1}-y_n\rangle - \frac{\mu}{\tau_{n+1}}\left(\|w_n-y_n\|^2 + \|x_{n+1}-y_n\|^2\right). \quad (24)$$

On the other hand, it follows from (5) that

$$f(y_n, y) \ge \tau_n f(y_n, x_{n+1}) + \tau_n\langle w_n - x_{n+1}, y - x_{n+1}\rangle,\ \forall y \in C. \quad (25)$$

Subtituting (24) into (25), we obtain

$$\begin{aligned} f(y_n, y) \ge\ & \frac{1}{2}\langle w_n - y_n, x_{n+1}-y_n\rangle - \frac{\mu}{2}\frac{\tau_n}{\tau_{n+1}}\left(\|w_n-y_n\|^2 + \|x_{n+1}-y_n\|^2\right) \\ & + \tau_n\langle w_n - x_{n+1}, y - x_{n+1}\rangle,\ \forall y \in C. \end{aligned} \quad (26)$$

By Lemma 3.1, we have

$$\lim_{n\to\infty}\tau_n \text{ exists and } \tau_n \ge \mathrm{m}\ \left\{\frac{\mu}{2\,\mathrm{m}\ \{c_1, c_2\}}, \tau_1\right\}.$$

This implies that the sequence $\{\tau_n\}$ is bounded. From (26), we get

$$\begin{aligned} f(y_{n_k}, y) \ge\ & \frac{1}{2}\langle w_{n_k} - y_{n_k}, x_{n_k+1}-y_{n_k}\rangle - \frac{\mu}{2}\frac{\tau_{n_k}}{\tau_{n_k+1}}\left(\|w_{n_k}-y_{n_k}\|^2 + \|x_{n+1}-y_{n_k}\|^2\right) \\ & + \tau_{n_k}\langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle,\ \forall y \in C. \end{aligned} \quad (27)$$

Letting $k \to +\infty$ in (27), we get

$$f(z^*, y) \ge 0,\ \forall y \in C.$$

This follows that $z^* \in EP(f,C)$.

Next, since (21) and the definition of $u = P_{EP(f,C)}(0)$, we have

$$\limsup_{k\to\infty}\langle -u, x_{n_k} - u\rangle = \langle -u, z^* - u\rangle \leq 0. \tag{28}$$

Combining (19) and (28), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle -u, x_{n_k+1} - u\rangle &\leq \limsup_{k\to\infty}\langle -u, x_{n_k} - u\rangle \\ &= \langle -u, z^* - u\rangle \\ &\leq 0. \end{aligned} \tag{29}$$

Hence, by (29), $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, Claim 3 and Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - u\| = 0$. That is the desired result.

4 Numerical Examples

In this section, we give some numerical examples to show the implementation of our proposed method. All computations are done in MATLAB R2016a and run on DELL $i-5$ Dual-Core 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let us consider a problem when the bifunction f is given as follows

$$f(x,y) := (Px + Qy + r)^T(y - x),$$

where $P = (p_{ij})_{N\times N}$ and $Q = (q_{ij})_{N\times N}$ are $N\times N$ symmetric positive semidefinite matrices such that $P - Q$ is also positive semidefinite and $r \in \mathbb{R}^N$. The bifunction f has the form of the one arising from a Nash-Cournot oligopolistic electricity market equilibrium model [6] and that f is convex in y, Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\| P - Q \|_2$, and the positive semidefinition of $P - Q$ implies that f is pseudomonotone. P and Q are matrices of the form A^TA with $A = (a_{ij})_{N\times N}$ being randomly generated in the interval $[-N, N]$.

Experiment 1

In this experiment, we consider feasible set

$$C := \prod_{i=1}^{N}[-10,10]$$

and compare our Algorithm 3.1 *with Algorithm* 3.1 *and Algorithm* 4.1 *proposed in Hieu et al. [11], and Algorithm 1 of [35] with different values of* $N = 5,10,20,30$. x_1 *is randomly generated in* $[-N, N]$ *except otherwise stated. The stopping criterion used is* $\|e_n\|_2 < \epsilon$ *with a tolerance* $\epsilon = 10^{-4}$.

Table 1: Methods Parameters Choice

Proposed Alg. 3.1	$\mu = 0.5$ $\epsilon = \frac{1}{n^2}$	$\tau = 0.01$ $x_0 = zeros(N,1)$	$\theta = 1$ $\|e_n\| = \|x_n - y_n\|$	$\alpha_n = \frac{1}{n}$
Hieu Alg. (3.1)	$\lambda_n = \frac{1}{1.01L^*}$	$\alpha_n = \frac{1}{(n+1)^{0.5}}$	$x_1, y_1 \in C$	$\|e_n\| = \|x_{n+1} - x_n\|$
Hieu Alg. (4.1)	$\lambda_0 = 1$ $x_1, y_0, y_1 \in C$	$\mu = 0.4$ $\|e_n\| = \|x_{n+1} - x_n\|$	$\alpha_n = \frac{1}{(n+1)^{0.9}}$	
Vuong Alg. (1)	$\lambda_n = \frac{1}{1.01L^*}$	$\alpha_n = \frac{1}{n}$	$\|e_n\| = \|x_n - y_n\|$	

Table 2: Comparison: Proposed Alg. 3.1 vs Hieu Alg. (3.1) vs Hieu Alg. (4.1) vs Vuong Alg.

N		Proposed Alg. 3.1	Hieu Alg. (3.1)	Hieu Alg. (4.1)	Vuong Alg.
5	No. of Iter.	5	234	60	1202
	CPU (Time)	3.4151	12.3607	3.0689	64.3597
10	No. of Iter.	3	110	269	84
	CPU (Time)	1.4410	6.1681	14.8703	4.6408
20	No. of Iter.	3	12	5317	60
	CPU (Time)	1.0064	0.81348	338.0696	3.57631
30	No. of Iter.	3	118	2427	26
	CPU (Time)	2.1625	7.7940	163.2326	1.6397

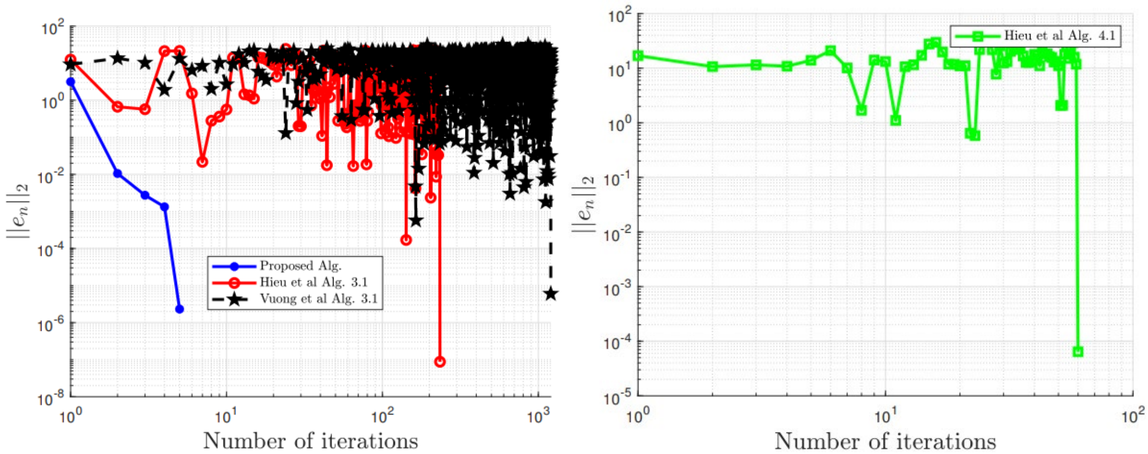

***Figure 1 Comparison: Proposed Alg. 3.1 vs Hieu Alg. (3.1) vs Hieu Alg. (4.1) vs Vuong** Alg.*
$N = 5$

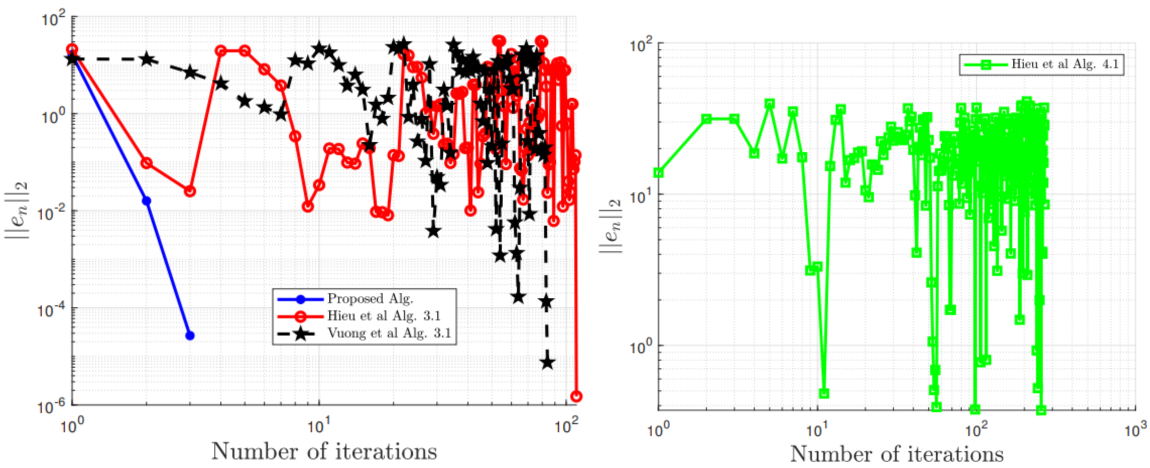

Figure 2 Comparison: Proposed Alg. 3.1 vs Hieu Alg. (3.1) vs Hieu Alg. (4.1) vs Vuong Alg.
$N = 10$

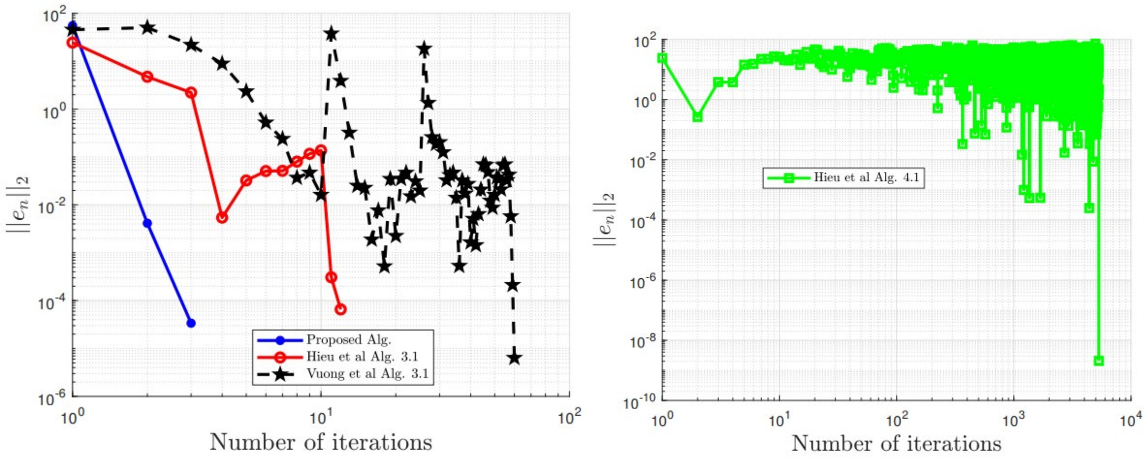

Figure 3 Comparison: Proposed Alg. 3.1 vs Hieu Alg. (3.1) vs Hieu Alg. (4.1) vs Vuong Alg.
$N = 20$

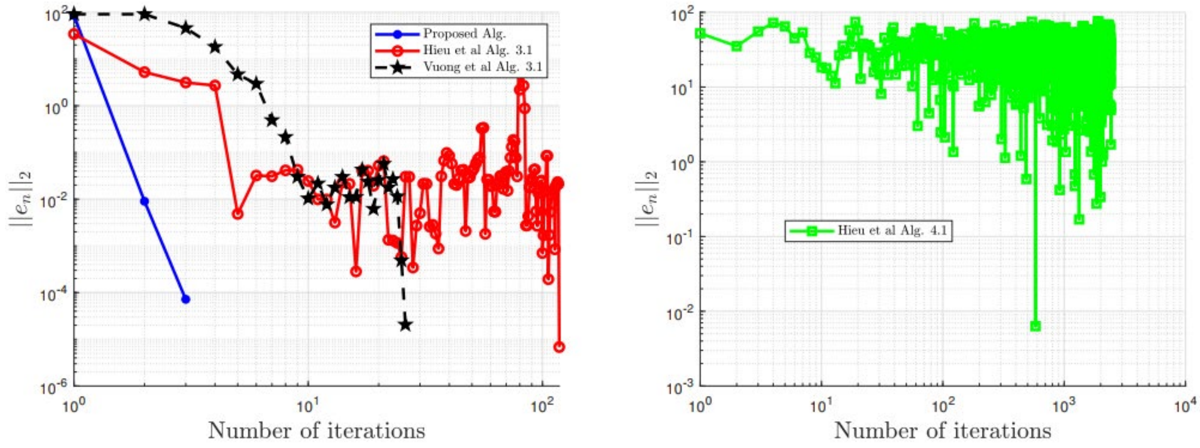

Figure 4 Comparison: Proposed Alg. 3.1 vs Hieu Alg. (3.1) vs Hieu Alg. (4.1) vs Vuong Alg.
$N = 30$

5. Conclusions

The paper has proposed a new modified subgradient extra-gradient method [4, 19] for approximating solutions of equilibrium problems in Hilbert spaces. The strong convergence theorem is established
under standard assumptions imposed on the equilibrium bifunctions. This work also studied the numerical behaviour of the proposed algorithm and compared it with the well known extra-gradient methods.

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