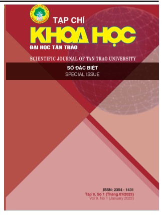
Vol 9. No 1_January 2023

TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

http://tckh.daihoctantrao.edu.vn/



CHEBYSHEV PSEUDOSPECTRAL METHOD FOR DUFFING NONLINEAR DIFFERENTIAL EQUATIONS

Le Anh Nhat

Tan Trao University, Viet Nam

Email address: leanhnhat@tuyenquang.edu.vn

DOI: 10.51453/2354-1431/2022/839

Article info

Received:25/9/2022

Revised: 20/10/2022

Accepted: 30/12/2022

Keywords:

Duffing oscillator, Duffing equation, pseudospectral methods, Duffing system, Chebyshev points.

Abstract:

The system of Duffing nonlinear differential equations is often used in dynamics, which are known to describe many important oscillating phenomena in nonlinear engineering systems. This article presents the pseudospectral method to calculate numerical solutions for nonlinear Duffing differential equations on the interval [–1, 1]. This method is based on the differential matrix using the Chebyshev Gauss – Lobatto points. To find numerical solutions of the nonlinear Duffing differential equations, we have built an iterative procedure. The software is used in calculating in this study is Mathematica 10.4. The obtained results show that this method has high accuracy with very small errors.

Vol 9. No 1_January 2023

TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

http://tckh.daihoctantrao.edu.vn/

PHƯƠNG PHÁP GIẢ PHỔ CHEBYSHEV CHO CÁC PHƯƠNG TRÌNH VI PHÂN PHI TUYẾN DUFFING

Lê Anh Nhật

Trường Đại học Tân Trào, Việt Nam

Địa chỉ email: leanhnhat@tuyenquang.edu.vn

DOI: 10.51453/2354-1431/2022/839

Thông tin bài viết

Ngày nhận bài: 25/9/2022
Ngày sửa bài: 20/10/2022
Ngày duyệt đăng: 30/12/2022

Từ khóa:

Dao động Duffing, phương trình Duffing, phương pháp giả phổ, hệ thống Duffing, điểm Chebyshev.

Tóm tắt

Hệ thống các phương trinhg vi phân phi tuyến Duffing thường được sử dụng trong động lực học, nó được biết đến để mô tả nhiều hiện tượng dao động quan trọng trong hệ thống kỹ thuật phi tuyến. Bài báo này trình bày phương pháp giả phổ để tính toán các nghiệm số cho phương trình vi phân phi tuyến Duffing trên khoảng [-1, 1]. Phương pháp này dựa trên ma trận vi phân sử dụng các điểm Chebyshev Gauss - Lobatto. Để tìm nghiệm số của các phương trình vi phân phi tuyến Duffing, chúng tôi đã xây dựng một thủ tục lặp. Phần mềm được sử dụng để tính toán trong nghiên cứu này là Mathematica 10.4. Kết quả số thu được cho thấy phương pháp này có độ chính xác cao và sai số rất nhỏ.

1. Introduction

The Duffing equation was known in 1918 in the article with the title "*Forced oscillations with variable natural frequency and their technical significance*" of George Duffing [1]. Since the appearance of the paper, there are many authors have been studied, expanded and developed the Duffing equation [2-6]. Simultaneously, many numerical methods also were studied to solve that equation force [4-7].

Consider the most general forced form of the Duffing nonlinear differential equation (or Duffing oscillator)

$$\frac{d^2}{dt^2}x(t)+\delta\frac{d}{dt}x(t)+\gamma x(t)+\beta x(t)^3=F\cos(\omega t),\quad a\le t\le b, \tag{1}$$

here, the numbers $\delta,\beta,\gamma,F,\omega$ are given constants, in which: $\delta\ge 0$ is controls the amount of damping; β – controls the amount of non-linearity in the restoring force; γ – controls the linear stiffness; F – is the amplitude of the periodic driving force; ω – is the angular frequency of the periodic driving force [4-6].

Depending on the choice of the γ and β, we had some the following special cases:

(i) The hard spring Duffing oscillator (H.S.D.O.)with $\gamma>0,\beta>0$;

(ii) The soft spring Duffing oscillator (S.S.D.O.) with $\gamma>0,\beta<0$;

(iii) The inverted Duffing oscillator (I.D.O.) with $\gamma < 0, \beta > 0$;

(iv) The nonharmonic Duffing oscillator (N.D.O.) with $\gamma = 0, \beta > 0$;

these special cases had been studied in the literature of Richard [5, Chapter 8].

Besides, depending on the parametric values δ, β, γ, F and ω, the problem (1) may be become other problems, for example:

If $\delta = 0$ and $F = 0$ then (1) becomes the cubic free undamped Duffing oscillator [2-3]

$$\frac{d^2}{dt^2}x(t)+\gamma x(t)+\beta x(t)^3=0,\quad a\le t\le b;$$

If $\gamma = 0$ and $\beta = \omega = 1$ then the equation (1) becomes the simplified Duffing oscillator

$$\frac{d^2}{dt^2}x(t)+\delta\frac{d}{dt}x(t)+x(t)^3=F\cos(t),\quad a\le t\le b,$$

this equation was studied by Ueda and so-called Ueda oscillator [4];

If $\gamma = -1$ and $\beta = 1$ then the equation (1) calls the Duffing-Holmes nonautonomous oscillator has form [8-10]

$$\frac{d^2}{dt^2}x(t)+\delta\frac{d}{dt}x(t)-x(t)+x(t)^3=F\cos(\omega t),\quad a\le t\le b.$$

Several numerical solutions have been studied so far dealing with the Duffing differential equation such as the modified differential transform method to obtain the approximate solutions of the nonlinear Duffing oscillator [11]; the collocation method is based on the radial basis functions to approximate the solution of the nonlinear controlled Duffing oscillator [12]; M. A. Al-Jawary proposed the Daftardar-Jafari method to solve the Duffing equations and to find the exact solution and numerical solutions [13]; M. Gorji-Bandpy applied Modified Homotopy Perturbation Method and the Max-Min approach to study the generalized Duffing equation [14]; in [15], the authors employed the new perturbation technique to solve strongly nonlinear Duffing oscillators; in [16], the authors used the Taylor Expansion to find approximate solution of Nonlinear Duffing Oscillator; to find numerical solution of the Duffing oscillator, the authors in [17-18] used the Legendre pseudospectral method, the authors in [19] used the spectral method, the authors in [20] used the Taylor matrix method; in [21], the authors proposed the post-verification method for solving the forced Duffing oscillator problems without prescribed periods; the analytical approximation technique basing on the energy balance method was used to determine approximate solutions for highly nonlinear Duffing oscillator [22-23]; the block multistep method is integrated with a variable order step size algorithm to find numerical solutions of the nonlinear Duffing oscillator [24].

This article uses the pseudospectral method based on Chebyshev differential matrix [25] to determine approximate solutions with the boundary conditions on the interval $[-1,1]$ take the form

$$x(-1)=\alpha,\ x(1)=\beta. \qquad ($$

2. Chebyshev differentiation matrix for Chebyshev Gauss – Lobatto points

For a fixed integer $N > 0$, set $J = \{0,1,2,...,N\}$, suppose that a grid function $v(x)$ is defined on the $N+1$ points Chebyshev Gauss – Lobatto. Choose the points $\{x_k, k\in J\}$ such that $x_k = \cos(k\pi/N)$, $k \in J$. They are the extrema of the N^{th} order in the Chebyshev polynomial $T_N(x)=\cos(N\cos^{-1}x)$.

The function $v(x)$ is interpolated by constructing the N^{th} order interpolation polynomial $g_j(x)$ such that $g_j(x_k)=\delta_{j,k}$,

$$p(x)=\sum_{j=0}^{n}p_j g_j(x) \qquad ($$

where $p(x)$ is the unique polynomial of degree N and $p_j = v(x_j)$, $j \in J$. The following can be shown:

$$g_j(x)=\frac{(-1)^{j+1}(1-x^2)T'_N(x)}{c_jN^2(x-x_j)},\quad j\in J,$$

where

$$c_j=\begin{cases}2, & j\in J_0,\\ 1, & j\in J_e\end{cases} \qquad ($$

with $J_e=\{1,2,...,N-1\}$ and $J_0=\{0,N\}$.

As we know the values of $p(x)$ at $N+1$ points, we would like to find approximately the values of the derivative of $p(x)$ at those points,

$p'(x)=\frac{d}{dx}p(x)$, in the matrix form $p'=D_C p$. The matrix $D_C=\left\{d_{i,j}^{(1)}\right\}$ is called the Chebyshev differentiation matrix.

Evidently, the derivative of $p(x_j)$ is

$$p'(x_j)=\sum_{k=0}^{N}(D_C)_{j,k}\,p(x_k),\quad j\in J.$$

We has the entries $d_{i,j}^{(1)}=g'_i(x_j)$, which are [25-29]

$$d_{0,0}^{(1)}=\frac{2N^2+1}{6}=-d_{N,N}^{(1)},$$

$$d_{i,i}^{(1)}=-\frac{x_i}{2(1-x_i^2)},\quad i\in J_e,$$

$$d_{i,j}^{(1)}=\frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i-x_j},\quad i\neq j,\quad i,j\in J,$$

where c_k is determined by the formula (8).

Similarly, $p'(x)$ is a polynomial of degree $N-1$ and the second-order differentiation matrix D_C^2. We have $p''(x)=\frac{d^2}{dx^2}p(x)$ or in the matrix form $p''=D_C^2p$, the second-order derivative of $p(x_j)$ becomes

$$p''(x_j)=\sum_{k=0}^{N}(D_C^2)_{j,k}\,p(x_k),\quad j\in J$$

with the entries $d_{i,j}^{(2)}$ are determined by the formula $D_C^2=\left(D_C\right)^2$, or they are identified as follows [30-31]:

$$d_{0,0}^{(2)}=d_{N,N}^{(2)}=\frac{N^4-1}{15}$$

$$d_{i,i}^{(2)}=-\frac{(N^2-1)(1-x_i^2)+3}{3(1-x_i^2)^2},\quad i\in J_e$$

$$d_{i,j}^{(2)}=\frac{(-1)^{i+j+1}}{c_j}\frac{2-x_ix_j-x_i^2}{\left(1-x_i^2\right)\left(x_i-x_j\right)^2},\quad i\neq j,\quad i,j\in J_e,$$

$$d_{0,j}^{(2)}=\frac{2}{3}\frac{(-1)^j}{c_j}\frac{\left(2N^2+1\right)\left(1-x_j\right)-6}{\left(1-x_j^2\right)},\quad j\in J$$

$$d_{N,j}^{(2)}=\frac{2}{3}\frac{(-1)^{j+N}}{c_j}\frac{\left(2N^2+1\right)\left(1+x_j\right)-6}{\left(1+x_j^2\right)},\quad j\in J.$$

The entries of the second-order differentiation matrix satisfy the identity $d_{j,k}^{(2)}=\sum_{i=0}^{N}d_{j,i}^{(1)}d_{i,k}^{(1)}$.

3. Chebyshev pseudospectral method Suppose that

$$\frac{d^2}{dx^2}x(t)=g(t),\quad t(-1)=\alpha_1,\quad t(1)=\alpha_2,$$

and the collocation points $\{t_i\}$ so that $-1=t_0<t_1<...<t_n=1$.

We know that

$$\frac{d^2}{dx^2}x_N(t_i)=\sum_{k=0}^{N}(D_C)_{i,k}^2x_N(t_k).$$

Therefore, problem (13) becomes

$$\sum_{k=0}^{N}(D_C^2)_{i,k}x_N(t_k)=g(t_i),\quad i\in J_e,\quad x_N(t_N)=\alpha_1,\quad x_N(t_0)=\alpha_2$$

Alternately, we partition the matrix D_C into matrices.

Consider the matrix D_C, we cut off the first-row $d_{0,i}^{(1)}$ and last-row $d_{N,i}^{(1)}$ with $i\in J_e$, then we partition that matrix into three matrices:

$$\underbrace{\begin{pmatrix} d_{1,0}^{(1)} \\ d_{2,0}^{(1)} \\ \vdots \\ d_{N-1,0}^{(1)}\end{pmatrix}}_{\tilde{e}_0^{(1)}}\underbrace{\begin{pmatrix} d_{1,1}^{(1)} & d_{1,2}^{(1)} & \dots & d_{1,N-1}^{(1)} \\ d_{2,1}^{(1)} & d_{2,2}^{(1)} & \dots & d_{2,N-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N-1,1}^{(1)} & d_{N-1,2}^{(1)} & \dots & d_{N-1,N-1}^{(1)}\end{pmatrix}}_{E^{(1)}}\underbrace{\begin{pmatrix} d_{1,N}^{(1)} \\ d_{2,N}^{(1)} \\ \vdots \\ d_{N-1,N}^{(1)}\end{pmatrix}}_{\tilde{e}_N^{(1)}}.$$

And we can rewrite in the matrix form $\tilde{e}_0^{(1)}=\left\{d_{i,0}^{(1)}\right\}$, $E^{(1)}=\left\{d_{i,j}^{(1)}\right\}$ and $\tilde{e}_N^{(1)}=\left\{d_{i,N}^{(1)}\right\}$ with $i,j\in J_e$ [32-33].

Similarly, we partition matrix D_C^2 and can rewrite in the matrix form: $\tilde{e}_0^{(2)}=\left\{d_{i,0}^{(2)}\right\}$, $E^{(2)}=\left\{d_{i,j}^{(2)}\right\}$ and $\tilde{e}_N^{(2)}=\left\{d_{i,N}^{(2)}\right\}$ with $i,j\in J_e$.

Thus the problem (13) can be written in the matrix form

$$\alpha_2 \tilde{e}_0^{(2)} + E^{(2)} x + \alpha_1 \tilde{e}_N^{(2)} = g,$$

where x and g denote the vectors

$$x = \begin{pmatrix} x_N(t_1) \\ x_N(t_2) \\ \vdots \\ x_N(t_{N-1}) \end{pmatrix}, \quad g = \begin{pmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_{N-1}) \end{pmatrix}.$$

Using the Chebyshev pseudospectral method to solve number problems (13), the simple second-order differential equation (13) can rewrite in the matrix form (17). Thence, we find numerical solution x. Section 4 and 5 will present its application for the Duffing nonlinear differential equations.

4. Applications

Consider the Duffing nonlinear differential equation

$$\frac{d^2}{dt^2} x(t) + \delta \frac{d}{dt} x(t) + \gamma x(t) + \beta x(t)^3 = F\cos(\omega t), \qquad -1 \le t \le 1, \tag{18}$$

with the boundary conditions $x(-1) = \alpha_1$ and $x(1) = \alpha_2$.

We apply the Section 3 to the equation (18), $\frac{d}{dt} x(t)$ can be written in the matrix form

$$\frac{d}{dt} x(t) = \alpha_2 \tilde{e}_0^{(1)} + E^{(1)} x + \alpha_1 \tilde{e}_N^{(1)}.$$

So, we can rewrite the equation (18) in the matrix form as follows:

$$\left(E^{(2)} + \delta E^{(1)} + P\right) x + \alpha_2 \left(\tilde{e}_0^{(2)} + \delta \tilde{e}_0^{(1)}\right) + \alpha_1 \left(\tilde{e}_N^{(2)} + \delta \tilde{e}_N^{(1)}\right) = Q, \tag{19}$$

where P denotes the diagonal matrice with elements $\left\{\gamma + \beta x(t_i)^2\right\}$ and Q denotes the vector with elements $\left\{F\cos(\omega t_i)\right\}$ with $i \in J_e$:

$$P = \begin{pmatrix} \gamma + \beta x(t_1)^2 & 0 & \dots & 0 \\ 0 & \gamma + \beta x(t_2)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma + \beta x(t_{N-1})^2 \end{pmatrix}, \quad \mathrm{Q} = \begin{pmatrix} F\cos(\omega t_1) \\ F\cos(\omega t_2) \\ \vdots \\ F\cos(\omega t_{N-1}) \end{pmatrix}.$$

Similarly, the cubic free undamped Duffing oscillator (2) can rewrite in the matrix form as follows:

$$\left(E^{(2)} + P_1\right) x + \alpha_2 \tilde{e}_0^{(2)} + \alpha_1 \tilde{e}_N^{(2)} = 0, \tag{20}$$

where P_1 denotes the diagonal matrice with elements $\left\{\gamma + \beta x(t_i)^2\right\}$ with $i \in J_e$.

The Ueda oscillator (3) in the matrix form:

$$\left(E^{(2)} + \delta E^{(1)} + P_2\right) x + \alpha_2 \left(\tilde{e}_0^{(2)} + \delta \tilde{e}_0^{(1)}\right) + \alpha_1 \left(\tilde{e}_N^{(2)} + \delta \tilde{e}_N^{(1)}\right) = Q_2, \tag{21}$$

where P_2 denotes the diagonal matrice with elements $\left\{x(t_i)^2\right\}$ and Q_2 denotes the vector with elements $\left\{F\cos(\omega t_i)\right\}$ with $i \in J_e$.

The matrix form of the Duffing – Holmes nonautonomous oscillator (4) as follows:

$$\left(E^{(2)} + \delta E^{(1)} + P_3\right) x + \alpha_2 \left(\tilde{e}_0^{(2)} + \delta \tilde{e}_0^{(1)}\right) + \alpha_1 \left(\tilde{e}_N^{(2)} + \delta \tilde{e}_N^{(1)}\right) = Q_3, \tag{22}$$

where P_3 denotes the diagonal matrice with elements $\{-1 + x(t_i)^2\}$ and Q_3 denotes the vector with elements $\{F\cos(\omega t_i)\}$ with $i \in J_e$.

To find the solution $x_N(t_i)$ of equations (19), (20), (21) and (22) we may be able to approach it with an iterative procedure has the following:

Procedure FindSolution;

Begin

set $u^{(\text{old})} := I^T$; $\varepsilon := 1$; $er = 10^{-8}$;

$Q = F\cos(wt_i)$;

while $\varepsilon > er$ **do**

Begin

$P := \gamma + \beta u^{(old)}$;

$T = E^{(2)} + \delta E^{(1)} + P$;

$u^{(new)} = T^{-1}Q$;

$\varepsilon := \left| Min\left\{u_1^{(new)} - u_1^{(old)},\ u_2^{(new)} - u_2^{(old)},\ \ldots,\ u_{n-1}^{(new)} - u_{n-1}^{(old)}\right\}\right|$;

$u^{(old)} := u^{(new)}$;

end;

return $u^{(old)}$;

End;

here I is the unit vector, er is the error that might change.

5. Numerical results

To calculate numerical results of Duffing nonlinear differential equations by Chebyshev pseudospectral method (CPSM), we use Mathematica version 10.4. We compare numerical results computed by CPSM and the numerical results computed by Mathematica's NDSolve.

In the numerical samples, for convenience, we shall restrict ourselves to the case $N = 100$, with $N > 100$ causes no difficulties in calculation. In equations (1), (3) and (4) we use the Dirichlet boundary conditions $x(-1) = 0$ and $x(1) = 0$. With the equation (2), we utilize the inhomogeneous boundary conditions, which mean that $x(-1) \neq 0$ and $x(1) \neq 0$.

In figures, dots illustrate numerical solutions of CPSM and solid lines illustrate numerical results of Mathematica's NDSolve. The Fig. 1 and 2 illustrate numerical solutions of equations (1) and (2) in cases H.S.D.O., S.S.D.O., I.D.O. and N.D.O. With the equation (1), we put $\delta = 1.2$, $F = 2$, $\omega = 2\pi$ and $(\gamma, \beta) = \{(0.7;0.5),(2;-3),(-0.7;0.5),(0;0.6)\}$. The equation (2), we put boundary conditions $x(-1) = x(1) = 0.1$ and $\{\gamma, \beta\} = \{(0.7,0.5);(2,-3);(-0.7,0.5);(0,0.6)\}$. Table 1 is the biggest odds between two numerical solutions calculated by CPSM and Mathematica's NDSolve.

Table 1. The biggest odds between two numerical solutions calculated by CPSM and Mathematica's NDSolve of equations (1) and (2).

Case	Equation (1)	Equation (2)
H.S.D.O.	2.98171×10^{-8}	1.99901×10^{-8}
S.S.D.O.	1.25345×10^{-8}	4.41022×10^{-8}
I.D.O.	2.98753×10^{-8}	2.70486×10^{-8}
N.D.O.	2.9722×10^{-8}	1.56076×10^{-8}

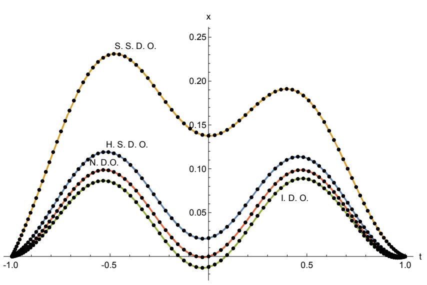


Fig. 1. The Duffing nonlinear differential equation (1)

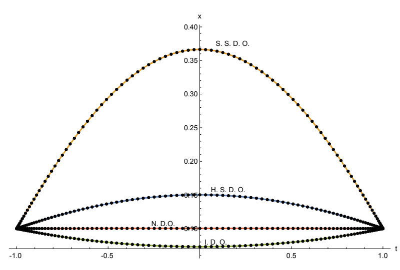


Fig. 2. The cubic free undamped Duffing oscillator (2)

Fig. 3 and 4. illustrate numerical solutions of the equations (3) and (4) with the Dirichlet boundary conditions. With the equation (3), we put $\delta = 10$ and $F = \{-4; -2; 2; 4\}$. The equation (4), we put $F = 2, \omega = 2\pi$ and $\delta = \{1.2; 3; 5.7; 9.5\}$. The biggest odds between two numerical solutions calculated by CPSM and Mathematica's NDSolve shown in Table 2.

Table 2. The biggest odds between two numerical solutions calculated by CPSM and Mathematica's NDSolve of equations (3) and (4)

Cases	**Equation (3)**	**Cases**	**Equation (4)**
F = -4	2.74712×10^{-8}	δ = 1.2	1.48026×10^{-7}
F = -2	1.56173×10^{-8}	δ = 3	6.10488×10^{-8}
F = 2	1.56194×10^{-8}	δ = 5.7	3.18598×10^{-8}
F = 4	2.75009×10^{-8}	δ = 9.5	3.45512×10^{-8}

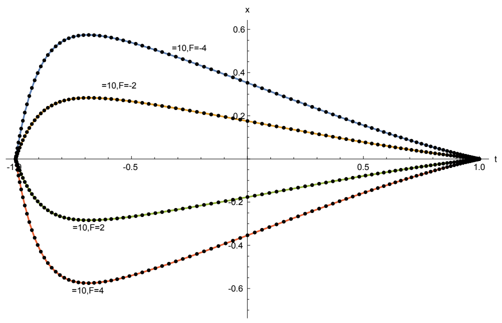


Fig. 3. The Ueda oscillator (3)

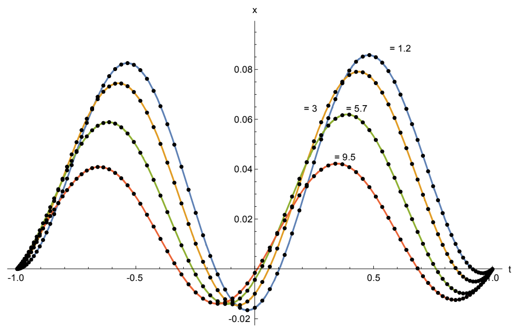


Fig. 4. The Duffing–Holmes nonautonomous oscillator (4)

Fig. 3 and 4. illustrate numerical solutions of the equations (3) and (4) with the Dirichlet boundary conditions. With the equation (3), we put $\delta = 10$ and $F = \{-4; -2; 2; 4\}$. The equation (4), we put $F = 2, \omega = 2\pi$ and $\delta = \{1.2; 3; 5.7; 9.5\}$. The biggest odds between two numerical solutions calculated by CPSM and Mathematica's NDSolve shown in Table 2.

The obtained results of the equations (1), (2), (3) and (4) shown in Table 1 and 2 show that this method has high accuracy with very small errors.

6. Conclusion

We present the pseudospectral method basing on the differentiation matrix using the Chebyshev Gauss – Lobatto points to calculate numerical solutions for nonlinear Duffing differential equations on the interval [–1, 1]. We use the iterative procedure to find numerical solutions of the Duffing nonlinear differential equations and consider four special cases of the Duffing differential equations system. The numerical results demonstrate the efficiency and of the reliable method for solving this problem.

REFERENCES

[1]. Kovacic I., Brennan M.J. (2011). *Background: On Georg Duffing and the Duffing equation, The Duffing Equation: Nonlinear Oscillators and their Behaviour*, Wiley.

[2]. Sibanda P., Khidir A. (2011). *A new modification of the HPM for the Duffing equation with cubic nonlinearity*, ICACM'11, Lanzarote, Spain, pp. 139–143. https://doi.org/10.24297/jap.v2i2.2099

[3]. Salas A.H., Castillo J. E. (2014). *Exact Solution to Duffing Equation and the Pendulum Equation*, Appl. Math. Sci., 8(176): 8781–8789. https://doi.org/10.12988/ams.2014.44243

[4]. Korsch H.J., Jodl H.-J., Hartmann T. (2008). *Chaos: A Program Collection for the PC*, Springer.

[5]. Enns, R.H., McGuire, G.C. (2004). *Forced Oscillators*. In: Nonlinear Physics with Mathematica for Scientists and Engineers. Birkhäuser, Boston, MA. https://doi.org/10.1007/978-1-4612-0211-0_8

[6]. Kovacic I., Brennan M.J. (2011). *The Duffing Equation: Nonlinear Oscillators and their Behaviour*, ed. first, Wiley.

[7]. Bashkirtseva, I.A. (2018). *The impact of colored noise on the equilibria of nonlinear dynamic systems*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 28(2): 133–142. https://doi.org/10.20537/vm180201

[8]. Longsuo L. (2011). *Suppressing Chaos of Duffing-Holmes System Using Random Phase*, Math. Probl. Eng., id. 538202, 8p. https://doi.org/10.1155/2011/5382028

[9]. Tamaseviius A., Bumelien S., Kirvaitis R. et al (2009). *Autonomous Duffing-Holmes Type Chaotic Oscillator*, Elektronika ir Elektrotechnika, 5(93): 43–46.

[10]. Tamaseviciute E., Tamasevicius A., Mykolaitis G. et al (2008). *Analogue Electrical Circuit for Simulation of the Duffing-Holmes Equation, Nonlinear Analysis: Modelling and Control*, 13(2): 241–252.

[11]. Nourazar S., Mirzabeigy A. (2013). *Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method*, Sci. Iran. B, 20(2): 364–368. https://doi.org/10.1016/j.scient.2013.02.023

[12]. Rad J.A., Kazem S., Parand K. (2012). *A numerical solution of the nonlinear controlled Duffing oscillator by radial basis functions*, Computers and Mathematics with Applications, 64(6): 2049–2065.

[13]. Majeed AL-Jawary, Sayl Abd-AL-Razaq (2016). *Analytic and numerical solution for duffing equations*, Int. J. Basic Appl. Sci., 5(2): 115–119. https://doi.org/10.14419/IJBAS.V5I2.5838

[14]. Gorji-Bandpy M., Azimi M., Mostofi M. (2011). *Analytical methods to a generalized Duffing oscillator*, Australian J. Basic Applied Sci., 5(11): 788–796.

[15]. El-Naggar A.M., Ismail G.M. (2016). *Analytical solution of strongly nonlinear Duffing oscillators*, Alex. Engg. J., 55(2): 1581–1585. https://doi.org/10.1016/j.aej.2015.07.017

[16]. A. Okasha El-Nady, Maha M.A. Lashin. (2016). *Approximate Solution of Nonlinear Duffing Oscillator Using Taylor Expansion*, J. Mech. Engi. Auto., 6(5): 110–116. https://doi.org/10.5923/j.jmea.20160605.03

[17]. Razzaghi M., Elnagar G.N. (1994). *Numerical solution of the controlled Duffing oscillator by the pseudospectral method*, J. Comput. Appl. Math., 56(3): 253–261. https://doi.org/10.1016/0377-0427(94)90081-7

[18]. Saadatmandi A, Mashhadi-Fini F. (2015). *A pseudospectral method for nonlinear Duffing equation involving both integral and non-integral forcing terms*, Math. Methods Appl. Sci., 38(7): 1265–1272. https://doi.org/10.1002/mma.3142

[19]. Elnagar G.N., Razzaghi M. (1997). *A Chebyshev spectral method for the solution of nonlinear optimal control problems*, Appl. Math. Modelling, 21(5): 255–260. https://doi.org/10.1016/S0307-904X(97)00013-9

[20]. Bulbul B., Sezer M. (2013). *Numerical Solution of Duffing Equation by Using an Improved Taylor Matrix Method*, J. Appl. Math., id. 691614, 7p. https://doi.org/10.1155/2013/691614

[21]. Hong-Yen Lin, Chien-Chang Yen, Kuo-Ching Jen, Kang C. Jea (2014). *A Postverification Method for Solving Forced Duffing Oscillator Problems without Prescribed Periods*, J. Appl. Math., id. 317640, 11p. https://doi.org/10.1155/2014/317460

[22]. Alal Hosen M., Chowdhury M.S.H., Yeakub Ali M., Faris Ismail A. (2017). *An analytical approximation technique for the duffing oscillator based on the energy balance method*, Ital. J. Pure Appl. Math., 37: 455–466. ISSN 1126-8042

[23]. Ganji D.D., Gorji M., Soleimani S. et al (2009). *Solution of nonlinear cubic-quintic Duffing oscillators using He's Energy Balance Method*, J Zhejiang Univ Sci A, 10(9): 1263–1268. https://doi.org/10.1631/jzus.A0820651

[24]. Rasedee A.F.N., Abdul Sathar M.H., Ishak N., et al (2017). *Solution for nonlinear Duffing oscillator using variable order variable stepsize block method*. MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics, 33(2): 165–176. https://doi.org/10.11113/matematika.v33.n2.1015

[25]. Mason J.C., Handscomb D.C. (2003). *Chebyshev Polynomials*, CRC Press LLC.

[26]. Trefethen L.N. (2000). *Spectral Methods in Matlab*, SIAM.

[27]. Don W.S., Solomonoff A. (1991). *Accuracy and Speed in Computing the Chebyshev Collocation Derivative*, SIAM J. of Sci. Comput., 16(6): 1253–1268. https://doi.org/10.1137/0916073

[28]. Tinuade Odeyemi, Abdolmajid Mohammadian, Ousmane Seidou (2012). *Application of the Chebyshev pseudospectral method to van der Waals fluids*, Commun Nonlinear Sci Numer Simulat, 17(9): 3499–3507. https://doi.org/10.1016/j.cnsns.2011.12.025

[29]. Jensen A. (2009). *Lecture Notes on Spectra and Pseudospectra of Matrices and Operators*, Aalborg University.

[30]. Dang-Vu H. (1995), *Delcarte C. Hopf Bifurcation and Strange Attractors in Chebyshev Spectral Solutions of the Burgers Equation*, Appl. Math. Comput., 73(2-3): 99–113. https://doi.org/10.1016/0096-3003(94)00242-8

[31]. Canuto C, Quarteroni A., Hussaini M.Y. et al (2006). *Spectral Methods: Fundamentals in Single Domains*, Springer–Verlag Berlin Heidelberg. https://doi.org/10.1007/978-3-540-30726-6

[32]. Nhat L.A. (2018). *Using differentiation matrices for pseudospectral method solve Duffing Oscillator*, J. Nonlinear Sci. Appl., 11(12): 1331–1336. http://doi.org/10.22436/jnsa.011.12.04

[33]. Nhat L.A. (2019). *Pseudospectral method for the second-order autonomous nonlinear differential equations*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 29(1): 61–72. https://doi.org/10.20537/vm190106