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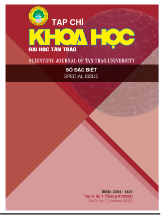

PULLBACK ATTRACTORS FOR A NON-AUTONOMOUS SEMILINEAR STRONGLY DEGENERATE PARABOLIC EQUATION

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Abstract:

In this paper, using the asymptotic a priori estimate method, we prove the existence of pullback attractors for a non-autonomous semilinear strongly degenerate parabolic equation, without restriction on the growth order of the polynomial type non-linearity and with a suitable exponential growth of the external force. The obtained results improve some recent ones for the non-autonomous reaction–diffusion equations.

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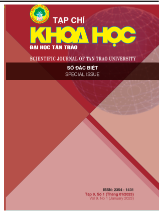

TẬP HÚT LÙI CHO LỚP PHƯƠNG TRÌNH PARABOLIC NỬA TUYẾN TÍNH SUY BIẾN MẠNH KHÔNG ÔTÔNÔM

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Từ khóa:
Phương trình parabolic suy biến, Toán tử Δ_λ, Tập hút lùi.

Tóm tắt:

Trong bài báo này, chúng tôi đã sử dụng phương pháp đánh giá tiệm cận để chứng minh sự tồn tại tập hút lùi cho lớp phương trình parabolic nửa tuyến tính suy biến mạnh không ôtônôm, với số hạng phi tuyến loại đa thức không hạn chế về bậc tăng trưởng và ngoại lực với mức tăng trưởng hàm mũ. Các kết quả thu được là sự mở rộng các kết quả gần đây cho lớp phương trình phản ứng khuếch tán không ôtônôm.

1 INTRODUCTION

In this paper, we consider the following non-autonamous semilinear strongly degenerate parabolic equation

$$\begin{cases} u_t - \Delta_\lambda u + f(u) = g(x,t), & x \in \Omega, t > \tau, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u\,|_{t=\tau} = u_\tau(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega, u_\tau \in L^2(\Omega)$, the nonlinear term $f(u)$ and the external fore g satisfy some conditions specified later, and Δ_λ is a strongly degenerate operator of the form

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}({\lambda_i}^2(x)\partial_{x_i}),$$

where $\lambda = (\lambda_1, \ldots, \lambda_N) : \mathbb{R}^N \to \mathbb{R}^N$ satisfies certain conditions specified below. This operator was introduced by Franchi and Lanconelli in [2] and recently reconsidered in [5] under an additional assumption that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$. Here the functions $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of class C^1 outside the coordinate hyperplanes, i.e., $\lambda_i > 0, i = 1, \ldots, N$ in $\mathbb{R}^N \setminus \prod$, where $\prod = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid \prod_{i=1}^{N} x_i =$

$0\}$. As in [5] we assume that λ_i satisfy the following properties:

1. $\lambda_1(x) \equiv 1$, $\lambda_i(x) = \lambda_i(x_1, \ldots, x_{i-1}), i = 2, \ldots, N$;

2. For every $x \in \mathbb{R}^N$, $\lambda_i(x) = \lambda_i(x^*), i = 1, \ldots, N$, where

$$x^* = (|x_1|, \ldots, |x_N|) \text{ if } x = (x_1, \ldots, x_N);$$

3. There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \rho \lambda_i(x) \quad \forall k \in \{1, \ldots, i-1\}, \quad i = 2, \ldots, N,$$

and for every $x \in \mathbb{R}^N_+ := \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_i \geq 0 \ \forall i = 1, \ldots, N\}$;

4. There exists a group of dilations $\{\delta_t\}_{t>0}$

$$\delta_t : \mathbb{R}^N \to \mathbb{R}^N, \delta_t(x) = \delta_t(x_1, \ldots, x_N) = (t^{\epsilon_1} x_1, \ldots, t^{\epsilon_N} x_N),$$

where $1 \leq \epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_N$, such that λ_i is δ_t-homogeneous of degree $\epsilon_i - 1$, i.e.,

$$\lambda_i(\delta_t(x)) = t^{\epsilon_i - 1} \lambda_i(x), \quad \forall x \in \mathbb{R}^N, t > 0, \quad i = 1, \ldots, N.$$

This implies that the operator Δ_λ is δ_t-homogeneous of degree two, i.e.,

$$\Delta_\lambda(u(\delta_t(x))) = t^2 (\Delta_\lambda u)(\delta_t(x)), \ \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $\{\delta_t\}_{t>0}$, i.e.,

$$Q := \epsilon_1 + \cdots + \epsilon_N.$$

The homogeneous dimension Q plays a crucial role, both in the geometry and the functional associated to the operator Δ_λ.

The Δ_λ-Laplace operator contains many degenerate elliptic operators such as the Grushin type operator

$$G_\alpha = \Delta_x + |x|^{2\alpha} \Delta_y, \quad \alpha > 0,$$

where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, and the strongly degenerate operator of the form

$$P_{\alpha,\beta} = \Delta_x + \Delta_y + |x|^{2x} |y|^{2y} \Delta_z,$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$ $(N_i \geq 1, i = 1, 2, 3), \alpha, \beta$ are real positive constants, see [17]. We refer the interested reader to [6, Section 2.3] for other examples of Δ_λ-Laplacians. See also [13] for recent results related to elliptic equations involving this operator.

In the last years, the existence and long-time behavior in terms of existence of global attractors of solutions to semilinear parabolic equations involving the above strongly degenerate operators have been studied extensively by a number of authors. Up to now, there are two main kinds of nonlinearities that have been considered. The first one is the class of nonlinearities that is locally Lipschitzian continuous and satisfies a Sobolev growth condition

$$|f(u) - f(v)| \leq C(1 + |u|^\rho + |v|^\rho)|u - v|, \qquad 0 \leq \rho \leq \frac{4}{Q-2},$$

and some suitable dissipative conditions; see [6, 7, 8, 17]. The second one is the class of nonlinearities that satisfies a polynomial growth

$$C_1|u|^p - C_0 \leq f(u)u \leq C_2|u|^p + C_0, \text{ for some } p \geq 2, \quad f'(u) \geq -\ell,$$

see [16, 17]. See also some related results in the case of bounded domains with stongly degeneracy operator Δ_λ (see [14]) and in the case of unbounded domains with stongly degeneracy operator $P_{\alpha,\beta}$ (see [18]), the more delicated case due to the lack of compactness of the Sobolev type embeddings.

Non-autonomous parabolic equations appear in many applications in natural sciences, so they are of great importance and interest. One way to study the long-time behavior of solutions of such equations is using the theory of pullback attractors. In the last few years, the existence of pullback attractors has been proved for many classes of parabolic equations [9, 10, 11, 12, 19, 20]. However, little seems to be known for non-autonomous semilinear strongly degenerate parabolic equations. This is the main motivation of the present paper. In this paper, using the asymptotic a priori estimate method, we prove the existence of pullback attractors for a non-autonomous semi-linear strongly degenerate parabolic equation involving the strongly degeneracy operator Δ_λ on the bounded domain $\Omega \subset \mathbb{R}^N$.

To study problem (1.1), we assume that the initial datum $u_\tau \in L^2(\Omega)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f:\mathbb{R}\to\mathbb{R}$ is a continuously differentiable function satisfying

$$f'(u) \geq -\ell, \quad (1.2)$$

$$C_1|u|^p - C_0 \leq f(u)u \leq C_2|u|^p + C_0, \quad p \geq 2 \quad (1.3)$$

where C_0, C_1, C_2, ℓ are positive constants and $F(u) = \int_0^u f(s)ds$ is a primitive of f and therefore by integrating by parts, we obtain

$$C_3|u|^p - C_4 \leq F(u) \leq C_5|u|^p + C_6 \quad \text{for all } u \in \mathbb{R}, \quad (1.4)$$

where C_3, C_4, C_5, C_6 are positive constants.

(G) $g \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\|g(t)\|^2_{L^2(\Omega)} \leq Ce^{\gamma t} \quad (1.5)$$

where C is positive constant, $\gamma < \gamma_1$ with $\gamma_1 > 0$ is the first eigenvalue of the operator $-\Delta_\lambda$ in Ω with the homogeneous Dirichlet boundary condition.

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some concepts and results on function spaces and pullback attractors which we will use. In Section 3, we prove the existence and uniqueness of weak solutions by utilizing the compactness method and weak convergence techniques in Orlicz spaces [4]. In Section 4, we construct the process associated to problem (1.1) and prove the existence of pullback attractor in space $L^2(\Omega)$.

2 PRELIMINARIES

2.1 Function space

To study problem (1.1), we use the weighted Sobolev space $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$ defined as the completion of $C^1_0(\Omega)$ in the norm

$$\|u\|_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} := \left(\int_\Omega |\nabla_\lambda u|^2 dx\right)^{1/2} = \|\nabla_\lambda u\|_{L^2(\Omega)}.$$

This is a Hilbert space with respect to the following scalar product

$$((u,v))_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} = \int_\Omega \nabla_\lambda u \cdot \nabla_\lambda v \, dx = (-\Delta_\lambda u, v), \quad \text{for all } u, v \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega).$$

By the result in [5], we know that the embedding $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

Let $\gamma_1 > 0$ be the first eigenvalue of the operator Δ_λ in Ω with homogeneous Dirichlet boundary conditions. Then

$$\gamma_1 = \inf\left\{\frac{\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \mid u \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)\backslash\{0\}\right\}.$$

Therefore,

$$\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} \geq \gamma_1 \|u\|^2_{L^2(\Omega)}, \quad (2.1)$$

for all $u \in \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$.

2.2 Pullback attractors

Let X be a Banach space. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X and $\|\cdot\|$ is the corresponding norm. For $A, B \subset X$, the Hausdorff semi-distance between A and B is defined by

$$\text{dist}(A,B) = \sup_{x\in A}\inf_{y\in B}\|x-y\|.$$

Let $\{U(t,\tau) : t \geq \tau, \tau \in \mathbb{R}\}$ be a process in X, i.e., a two-parameter family of mappings $U(t,\tau) : X \to X$ such that $U(\tau,\tau) = Id$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau, \tau \in \mathbb{R}$. The process $\{U(t,\tau)\}$ is said to be norm-to-weak continuous if $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$, as $x_n \to x$ in X, for all $t \geq \tau, \tau \in \mathbb{R}$. The following result is useful for verifying that a process is norm-to-weak continuous.

Proposition 2.1. [9] Let X, Y be two Banach spaces, X^*, Y^* be respectively their dual spaces. Assume that X is dense in Y, the injection $i : X \to Y$ is continuous, its adjoint $i^* : Y^* \to X^*$ is dense, and $\{U(t,\tau)\}$ is a continuous or weakly continuous process on Y. Then $\{U(t,\tau)\}$ is norm-to-weak continuous on X if and only if for $t \geq \tau$, $\tau \in \mathbb{R}$, $U(t,\tau)$ maps compact sets of X to be bounded sets of X.

Definition 2.1. [9] The process $\{U(t,\tau)\}$ is said to be pullback asymptotically compact if for any $t \in \mathbb{R}$, any $D \in \mathcal{B}(X)$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Definition 2.2. A process $\{U(t,\tau)\}$ is called pullback ω-limit compact if for any $\varepsilon > 0$, any $t \in \mathbb{R}$, and $D \in \mathcal{B}(X)$, there exists a $\tau_0(D,\varepsilon,t) \leq t$ such that

$$\alpha\Big(\bigcup_{\tau\leq\tau_0} U(t,\tau)D\Big) \leq \varepsilon,$$

where α is the Kuratowski measure of noncompactness of $B \in \mathcal{B}(X)$,

$$\alpha(B) = \inf\{\delta > 0 \mid B \text{ has a finite open cover of sets of diameter } \leq \delta\}.$$

Lemma 2.1. [9] A process $\{U(t,\tau)\}$ is pullback asymptotically compact if and only if it is pullback ω-limit compact.

Definition 2.3. A family of bounded sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$ is called pullback absorbing for the process $\{U(t,\tau)\}$ if for any $t \in \mathbb{R}$, any $D \in \mathcal{B}(X)$, there exists $\tau_0 = \tau_0(D,t) \leq t$ such that

$$\bigcup_{\tau \leq \tau_0} U(t,\tau)D \subset B(t).$$

Definition 2.4. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a pullback attractor for the process $\{U(t,\tau)\}$ if the following conditions hold:

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;

(2) $\hat{\mathcal{A}}$ is invariant, i.e.,

$$U(t,\tau)A(\tau) = A(t), \text{ for all } t \geq \tau;$$

(3) $\hat{\mathcal{A}}$ is pullback attracting, i.e.,

$$\lim_{\tau \to -\infty} dist(U(t,\tau)D, A(t)) = 0,$$

for all $D \in \mathcal{B}(X)$, and all $t \in \mathbb{R}$;

(4) if $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets then $A(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Theorem 2.2. [9] Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}$ is pullback asymptotically compact. If there exists a family of pullback absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\}$, then $\{U(t,\tau)\}$ has a unique pullback attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and

$$A(t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau)B(\tau)}.$$

3 EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

Definition 3.1. A function u is called a weak solution of problem (1.1) on (τ,T) if $u \in L^2(\tau,T;\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$, $f(u) \in L^{p'}(\tau,T;L^{p'}(\Omega)), u\,|_{t=\tau} = u_\tau$, $\frac{du}{dt} \in L^2(\tau,T;(\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega))^*) + L^{p'}(\tau,T;L^{p'}(\Omega))$ and

$$\int_\tau^T \int_\Omega \left(\frac{\partial u}{\partial t} w - \Delta_\lambda u w + f(u)w\right) dxdt = \int_\tau^T \int_\Omega g w dxdt,$$

for all test functions $w \in W := \overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega) \cap L^p(\tau,T;L^p(\Omega))$. Here, $(\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega))^*$ is the dual space of $\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)$ and $L^{p'}(\tau,T;L^{p'}(\Omega))$ is the dual space of $L^p(\tau,T;L^p(\Omega))$.

Theorem 3.1. Assume **(F)-(G)** hold. Then for any $u_\tau \in L^2(\Omega)$ and $T > \tau$ given, problem (1.1) has a unique weak solution u on the interval (τ,T). Moreover, the mapping $u_\tau \mapsto u(t)$ is continuous on $L^2(\Omega)$, that is, the solutions depend continuously on the initial data u_τ at time τ.

Chứng minh. i) Existence. We will prove the existence of a weak solution by using the compactness method. Let $\{u_n\}$ be the Galerkin appropriate solutions $u_n(t)$ in the form.

$$u_n(t) = \sum_{k=1}^n u_{nk}(t)e_k,$$

where $\{e_j\}_{j=1}^\infty$ are eigenvectors of the operator $-\Delta_\lambda$. We get u_n from solving the problem

$$\begin{cases} \left\langle \frac{\partial u_n}{\partial t}, e_k\right\rangle - \langle \Delta_\lambda u_n, e_k\rangle + \langle f(u_n), e_k\rangle = \langle g(x,t), e_k\rangle, \\ \big(u_n(\tau), e_k\big) = \big(u_\tau, e_k\big), k = 1,...,n. \end{cases}$$

Using the Peano theorem, we get the local existence of u_n. We now establish some a priori estimates for u_n. We have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + \int_\Omega f(u_n)u_n dx = \int_\Omega g(x,t)u_n dx. \quad (3.1)$$

By (1.3) and the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n(t)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + C_1\|u_n(t)\|^p_{L^p(\Omega)} - C_0|\Omega| \leq \frac{1}{2\gamma_1}\|g(x,t)\|^2_{L^2(\Omega)} + \frac{\gamma_1}{2}\|u_n(t)\|^2_{L^2(\Omega)}.$$

Using the inequality (2.1), we get

$$\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n(t)\|^2_{\overset{\circ}{W}{}^{1,2}_{\lambda}(\Omega)} + 2C_1\|u_n(t)\|^p_{L^p(\Omega)} \leq \frac{1}{\gamma_1}\|g(x,t)\|^2_{L^2(\Omega)} + 2C_0|\Omega|,$$

Integrating from τ to $t, 0 \le t \le T$, we get

$$\|u_n(t)\|^2_{L^2(\Omega)} + \int_\tau^t \|u_n(s)\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} ds$$
$$+2C_1 \int_\tau^t \|u_n(s)\|^p_{L^p(\Omega)} ds$$
$$\le \frac{1}{\gamma}\int_\tau^t \|g(x,s)\|^2_{L^2(\Omega)} ds + 2C_0|\Omega|(t-\tau) + \|u_n(\tau)\|^2_{L^2(\Omega)}.$$

This inequality yields

$$\{u_n\} \text{ is bounded in } L^\infty(\tau,T;L^2(\Omega)),$$
$$\{u_n\} \text{ is bounded in } L^p(\tau,T;L^p(\Omega)),$$
$$\{u_n\} \text{ is bounded in } L^2(\tau,T;\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)).$$

Due to the boundedness of $\{u_n\}$ in $L^2(\tau,T;\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))$, it is easy to check that $\{\Delta_\lambda u_n\}$ is bounded in $L^2(\tau,T;(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*)$. From the above results, we can assume that (up to a subsequence)

$$u_n \rightharpoonup u \text{ in } L^2(\tau,T;\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)),$$
$$u_n \rightharpoonup^* u \text{ in } L^\infty(\tau,T;L^2(\Omega)),$$
$$\Delta_\lambda u_n \rightharpoonup \Delta_\lambda u \text{ in } L^2(\tau,T;(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*).$$

up to a subsequence. By rewriting the equation as

$$\frac{du_n}{dt} = \Delta_\lambda u_n - f(u_n) + g(x,t),$$

we deduce that $\{\frac{du_n}{dt}\}$ is bounded in $L^2(\tau,T;(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^*)+L^{p'}(\tau,T;L^{p'}(\Omega))$, and therefore in $L^{p'}(\tau,T;(\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^* + L^{p'}(\Omega))$. Because $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \subset\subset L^2(\Omega) \subset (\overset{\circ}{W}{}^{1,2}_\lambda(\Omega))^* + L^{p'}(\Omega)$, by the Aubin-Lions-Simon compactness lemma (see [1]), we have that $\{u_n\}$ is compact in $L^2(\tau,T;L^2(\Omega))$. Hence, we may assume, up to a subsequence, that $u_n \to u$ a.e. in $\Omega \times [\tau,T]$. Since f is continuous it follows that $f(u_n) \to f(u)$ a.e. in $\Omega \times [\tau,T]$. Applying Lemma 6.1 in [3], we obtain

$$f(u_n) \rightharpoonup f(u) \quad \text{in } L^{p'}(\tau,T;L^{p'}(\Omega)).$$

Thus, u satisfies (3.1). It remains to show that $u(\tau) = u_\tau$. Choosing some test function $\varphi \in C_0^\infty([\tau,T];\overset{\circ}{W}{}^{1,2}_\lambda(\Omega) \cap L^p(\Omega)$ with $\varphi(T) = 0$, so in the 'limiting equation' one can integrate by parts in the t variable to give

$$\int_\tau^T -(u,\varphi')dt + \int_\tau^T \int_\Omega \nabla_\lambda u \nabla_\lambda \varphi dxdt$$
$$+\int_\tau^T \int_\Omega \big(f(u) - g(x,t)\big)\varphi dxdt = \big(u(\tau),\varphi(\tau)\big).$$

By applying the same procedure to the Galerkin approximations, we get that

$$\int_\tau^T -(u_n,\varphi')dt + \int_\tau^T \int_\Omega \nabla_\lambda u_n \nabla_\lambda \varphi dxdt$$
$$+\int_\tau^T \int_\Omega \big(f(u_n) - g(x,t)\big)\varphi dxdt$$
$$=\big(u_n(\tau),\varphi(\tau)\big).$$

Taking limits as $n \to \infty$ we conclude that

$$\int_\tau^T -(u,\varphi')dt + \int_\tau^T \int_\Omega \nabla_\lambda u \nabla_\lambda \varphi dxdt$$
$$+\int_\tau^T \int_\Omega \big(f(u) - g(x,t)\big)\varphi dxdt = \big(u_\tau,\varphi(\tau)\big).$$

since $u_n(\tau) \to u_\tau$. Thus, $u(\tau) = u_\tau$ and this implies that u is a weak solution to problem (1.1).

ii) Uniqueness and continuous dependence on the initial data. Let u and v be two weak solutions of (1.1) with initial data $u_\tau, v_\tau \in L^2(\Omega)$. Putting $w = u - v$, we have

$$\begin{cases} w_t - \Delta_\lambda w + \widetilde{f}(u) - \widetilde{f}(v) - \ell w = 0, \\ w(0) = u_0 - v_0, \end{cases} \tag{3.2}$$

where $\widetilde{f}(s) = f(s) + \ell s$. We choose $w(t)$ as a test function as in [4]. Consequently, the proof will be more involved.

We use some ideas in [4]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function

$$B_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \le k, \\ -k & \text{if } s < -k. \end{cases}$$

Consider the corresponding Nemytskii mapping $\widehat{B}_k : W \to W$ defined as follows

$$\widehat{B}_k(w)(x) = B_k(w(x)) \quad \text{for all } x \in \Omega.$$

By Lemma 2.3 in [4], we have that $\|\widehat{B}_k(w) - w\|_W \to 0$ as $k \to \infty$. Now multiplying the first equation in (3.2) by $\widehat{B}_k(w)$, then integrating over $\Omega \times (\varepsilon, t)$, where $t \in (\tau, T)$, we get

$$\int_\varepsilon^t \int_\Omega \frac{d}{ds}\big(w(s)\widehat{B}_k(w)(s)\big)dxds$$
$$-\int_\varepsilon^t \int_\Omega w\frac{d}{ds}\big(\widehat{B}_k(w)(s)\big)dxds$$
$$+\frac{1}{2}\int_\varepsilon^t \int_{x\in\Omega:|w(x,s)|\le k} |\nabla_\lambda w|^2 dxds$$
$$+\int_\varepsilon^t \int_\Omega \big(\widetilde{f}(u) - \widetilde{f}(v)\big)\widehat{B}_k(w)dxds$$
$$-\ell\int_\varepsilon^t \int_\Omega w\widehat{B}_k(w)dxds = 0.$$

Noting that $w\frac{d}{dt}\big(\widehat{B}_k(w)\big) = \frac{1}{2}\frac{d}{dt}\big((\widehat{B}_k(w))^2\big)$, we have

$$\begin{aligned}&\int_\Omega w(t)\widehat{B}_k(w)(t)dx - \frac{1}{2}\|\widehat{B}_k(w)(t)\|^2_{L^2(\Omega)}\\&+\frac{1}{2}\int_\varepsilon^t\int_{x\in\Omega:|w(x,s)|\le k}|\nabla_\lambda w|^2dxds\\&+\int_\varepsilon^t\int_\Omega \widetilde{f}'(\xi)w\widehat{B}_k(w)dxds\\&=\int_\Omega w(\varepsilon)\widehat{B}_k(w)(\varepsilon)dx - \frac{1}{2}\|\widehat{B}_k(w)(\varepsilon)\|^2_{L^2(\Omega)}\\&+\ell\int_\varepsilon^t\int_\Omega w\widehat{B}_k(w)dxds.\end{aligned}$$

Note that $\widehat{f}'(s)\ge 0$ and $sB_k(s)\ge 0$ for all $s\in\mathbb{R}$, by letting $\varepsilon\to\tau$ and $k\to\infty$ in the above equality, we obtain

$$\|w(t)\|^2_{L^2(\Omega)} \le \|w(\tau)\|^2_{L^2(\Omega)} + 2\ell\int_\tau^t\|w(s)\|^2_{L^2(\Omega)}ds.$$

Hence, by the Gronwall inequality of integral form, we get

$$\|w(t)\|^2_{L^2(\Omega)} \le \|w(\tau)\|^2_{L^2(\Omega)}e^{2\ell t} \le \|w(\tau)\|^2_{L^2(\Omega)}e^{2\ell T},$$

for all $t\in[\tau,T]$. Note that $w\in C([\tau,T];L^2(\Omega))$, in particular, we get the uniqueness if $w(\tau)=0$. □

4 EXISTENCE OF FULLBACK ATTRACTOR

Due to the results of Theorem 3.1, we can define a process

$$U(t,\tau): L^2(\Omega)\to \overset{\circ}{W}{}^{1,2}_\lambda(\Omega)\cap L^p(\Omega)$$

where $U(t,\tau)u_\tau = u(t)$ is the unique weak solution of (1.1) with the initial data u_τ at time τ. We will prove that the process $\{U(t,\tau)\}_{t>\tau}$ has a pullback attractor $\mathcal{A}$ in the space $L^2(\Omega)$.

For brevity, in the following lemmas, we give some formal calculation, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [15].

Lemma 4.1. The process $\{U(t,\tau)\}_{t\ge\tau}$ has a family of bounded pullback absorbing sets in $L^2(\Omega)$.

Chứng minh. Multiplying the first equation in (1.1) by u, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + \int_\Omega f(u)udx\\&=\int_\Omega g(x,t)udx.\end{aligned}\tag{4.1}$$

Using inequalities (1.3), (2.1), and the Cauchy inequality, we arrive at

$$\begin{aligned}&\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \gamma_1\|u\|^2_{L^2(\Omega)} \le 2C_0|\Omega|\\&+\frac{1}{\gamma_1}\|g(x,t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence, thanks to the Gronwall inequality, we obtain

$$\begin{aligned}\|u(t)\|^2_{L^2(\Omega)} \le&\|u_\tau\|^2_{L^2(\Omega)}e^{-\gamma_1(t-\tau)} + \frac{2C_0}{\gamma_1}|\Omega|\\&+\frac{e^{-\gamma_1 t}}{\gamma_1}\int_{-\infty}^t e^{\gamma_1 s}\|g(s)\|^2_{L^2(\Omega)}ds.\end{aligned}\tag{4.2}$$

This completes the proof. □

Lemma 4.2. The process $\{U(t,\tau)\}_{t\ge\tau}$ has a family of bounded pullback absorbing sets in $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)$.

Chứng minh. Multiplying the first equation in (1.1) by u_t and integrating by parts, we obtain

$$\begin{aligned}&\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + 2\int_\Omega F(u)dx\right)\\&=\int_\Omega g(x,t)u_tdx\\&\le\frac{1}{2}\|g(x,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)},\end{aligned}$$

thus

$$\frac{d}{dt}\left(\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + 2\int_\Omega F(u)dx\right) \le \|g(x,t)\|^2_{L^2(\Omega)}.\tag{4.3}$$

On the other hand, using (4.1) and inequalities (1.3), (2.1), we have

$$\begin{aligned}&\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + 2C_1\|u\|^p_{L^p(\Omega)}\\&\le 2C_0|\Omega| + \frac{1}{\gamma_1}\|g(x,t)\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.4}$$

Using (1.4), we obtain

$$\begin{aligned}&\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + C_7\int_\Omega F(u)dx\\&\le\frac{1}{\gamma_1}\|g(x,t)\|^2_{L^2(\Omega)} + C_8.\end{aligned}\tag{4.5}$$

Combining (4.3) and (4.5), we have

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + 2\int_\Omega F(u)dx\right)\\&+C_9\left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)} + 2\int_\Omega F(u)dx\right)\\&\le C_{10}\|g(x,t)\|^2_{L^2(\Omega)} + C_{11}.\end{aligned}\tag{4.6}$$

Multiplying (4.6)) by $(t-\tau)e^{\gamma_1 t}$, we get

$$\begin{aligned}&\frac{d}{dt}\Big[(t-\tau)e^{\gamma_1 t}\Big(\|u\|^2_{L^2(\Omega)}+\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}\\&+2\int_\Omega F(u)dx\Big)\Big]\\&\le\big(1+(\gamma_1-C_9)(t-\tau)\big)\Big(\|u\|^2_{L^2(\Omega)}+\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}\\&+2\int_\Omega F(u)dx\Big)\\&+\big(C_{11}+C_{10}\|g(x,t)\|^2_{L^2(\Omega)}\big)(t-\tau)e^{\gamma_1 t}.\end{aligned}$$

Integrating from τ to t, we get

$$\begin{aligned}&(t-\tau)e^{\gamma_1 t}\Big(\|u\|^2_{L^2(\Omega)}+\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}+2\int_\Omega F(u)dx\Big)\\&\le\big(1+C_{12}(t-\tau)\big)\int_\tau^t e^{\gamma_1 s}\Big(\|u\|^2_{L^2(\Omega)}+\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}\\&+2\int_\Omega F(u)dx\Big)ds+C_{13}(t-\tau)e^{\gamma_1 t}\\&+C_{10}(t-\tau)\int_\tau^t e^{\gamma_1 s}\|g(x,s)\|^2_{L^2(\Omega)}ds. \qquad (4.7)\end{aligned}$$

On the other hand, multiplying (4.2) by $e^{\gamma_1 t}$ and integrating from τ to t, we get

$$\begin{aligned}&\int_\tau^t e^{\gamma_1 s}\|u(t)\|^2_{L^2(\Omega)}ds\le(t-\tau)e^{\gamma_1\tau}\|u_\tau\|^2_{L^2(\Omega)}\\&+\frac{2C_0}{\gamma_1^2}|\Omega|e^{\gamma_1 t}+\frac{1}{\gamma_1}\int_{-\infty}^t\int_{-\infty}^s e^{\gamma_1 r}\|g(r)\|^2_{L^2(\Omega)}drds. \qquad (4.8)\end{aligned}$$

Multiplying (4.4)) by $e^{\gamma_1 t}$, we have

$$\begin{aligned}&\frac{d}{dt}\big(e^{\gamma_1 t}\|u\|^2_{L^2(\Omega)}\big)+e^{\gamma_1 t}\big(\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}+2C_1\|u\|^p_{L^p(\Omega)}\big)\\&\le\gamma_1 e^{\gamma_1 t}\|u\|^2_{L^2(\Omega)}+2C_0|\Omega|e^{\gamma_1 t}+\frac{e^{\gamma_1 t}}{\gamma_1}\|g(x,t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Integrating from τ to t and using (4.8), we get

$$\begin{aligned}&\int_\tau^t e^{\gamma_1 s}\big(\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}+2C_1\|u\|^p_{L^p(\Omega)}\big)ds\\&\le\big(1+\gamma_1(t-\tau)\big)e^{\gamma_1\tau}\|u_\tau\|^2_{L^2(\Omega)}+\frac{4C_0}{\gamma_1}|\Omega|e^{\gamma_1 t}\\&+\frac{1}{\gamma_1}\int_{-\infty}^t e^{\gamma_1 s}\|g(x,s)\|^2_{L^2(\Omega)}ds\\&+\int_{-\infty}^t\int_{-\infty}^s e^{\gamma_1 r}\|g(x,r)\|^2_{L^2(\Omega)}drds. \qquad (4.9)\end{aligned}$$

Combining (4.8) and (4.9), we get

$$\begin{aligned}&\int_\tau^t e^{\gamma_1 s}\big(\|u\|^2_{L^2(\Omega)}+\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}+2C_1\|u\|^p_{L^p(\Omega)}\big)ds\\&\le\big(1+(\gamma_1+1)(t-\tau)\big)e^{\gamma_1\tau}\|u_\tau\|^2_{L^2(\Omega)}\\&+\frac{2C_0(2\gamma_1+1)}{\gamma_1^2}|\Omega|e^{\gamma_1 t}+\frac{1}{\gamma_1}\int_{-\infty}^t e^{\gamma_1 s}\|g(x,s)\|^2_{L^2(\Omega)}ds\\&+\big(1+\frac{1}{\gamma_1}\big)\int_{-\infty}^t\int_{-\infty}^s e^{\gamma_1 r}\|g(x,r)\|^2_{L^2(\Omega)}drds. \qquad (4.10)\end{aligned}$$

Combining (4.7), (4.10) and using (1.4), we obtain

$$\begin{aligned}&\|u\|^2_{L^2(\Omega)}+\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}+2\int_\Omega F(u)dx\\&\le C\Big[\big(1+(t-\tau)+\frac{1}{t-\tau}\big)e^{-\gamma_1(t-\tau)}\|u_\tau\|^2_{L^2(\Omega)}\\&+\big(1+\frac{1}{t-\tau}\big)e^{-\gamma_1 t}\int_{-\infty}^t e^{\gamma_1 s}\|g(x,s)\|^2_{L^2(\Omega)}ds\\&+\big(1+\frac{1}{t-\tau}\big)e^{-\gamma_1 t}\int_{-\infty}^t\int_{-\infty}^s e^{\gamma_1 r}\|g(x,r)\|^2_{L^2(\Omega)}drds\Big].\end{aligned}$$

By (1.5), we have

$$\int_{-\infty}^t e^{\gamma_1 s}\|g(x,s)\|^2_{L^2(\Omega)}ds<\infty$$
$$\text{and}\int_{-\infty}^t\int_{-\infty}^s e^{\gamma_1 r}\|g(x,r)\|^2_{L^2(\Omega)}drds<\infty,$$

for all $t\in\mathbb{R}$. Hence, for every $t\in\mathbb{R}$ and bounded subset $\mathcal{B}$ in $L^2(\Omega)$, there exists a number $r_0=r_0(t)>0$ and $\tau_0=\tau_0(t,\mathcal{B})<t$ such that for all $\tau\le\tau_0, u_\tau\in\mathcal{B}$, we have

$$\|u\|^2_{\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)}\le r_0(t)\quad\text{for all}\quad\tau\le\tau_0,$$

with

$$\begin{aligned}r_0(t)=&2C\big(1+e^{-\gamma_1 t}\int_{-\infty}^t e^{\gamma_1 s}\|g(x,s)\|^2_{L^2(\Omega)}ds\\&+e^{-\gamma_1 t}\int_{-\infty}^t\int_{-\infty}^s e^{\gamma_1 r}\|g(x,r)\|^2_{L^2(\Omega)}drds\big).\end{aligned}$$

This completes the proof. □

As a direct consequence of Lemma 4.2 and the compactness of the embedding $\overset{\circ}{W}{}^{1,2}_\lambda(\Omega)\hookrightarrow L^2(\Omega)$, we get the main result of this section.

Theorem 4.1. Suppose **(F)**–**(G)** hold. Then the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by problem (1.1) has a pullback attractor in the space $L^2(\Omega)$.

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