



A RESULT OF MEAN SQUARE EXPONENTIAL STABILITY FOR DIFFERENTIAL DELAY EQUATIONS WITH STOCHASTIC NOISE.

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Abstract:

In the present paper, we aim to study the stability of a class of nonlinear differential equations with stochastic noise. We prove the existence of a unique global solution and show that the stochastic system under consideration has a unique global solution. Additionally, we also investigate the exponential stability in the mean square of the solution.



MỘT KẾT QUẢ VỀ TÍNH ỔN ĐỊNH MŨ THEO NGHĨA BÌNH PHƯƠNG TRUNG BÌNH ĐỐI VỚI LỚP PHƯƠNG TRÌNH VI PHÂN CÓ TRỄ VỚI NHIỀU NGẪU NHIÊN

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Tóm tắt:

Trong bài báo này chúng tôi nghiên cứu sự tồn tại nghiệm và tính ổn định của một lớp các phương trình vi phân phi tuyến với nhiễu ngẫu nhiên. Chứng minh bài toán có nghiệm toàn cục duy nhất và nghiên cứu sự ổn định mũ của nghiệm theo nghĩa bình phương trung bình.

Từ khóa:

Phương trình vi phân ngẫu nhiên; ổn định mũ bình phương trung bình; nhiễu ngẫu nhiên

1 INTRODUCTION

As is well known, stochastic differential equations (SDEs) have come to play an important role in many branches of science and industry, such as biology, physics, economics, engineering and financial market. However, these applications depend heavily on the stability to a great extent. Hence, the stability of SDEs has received a great deal of attention over the past years, and many related results have appeared in the literature, e.g. see [1, 2, 3, 9, 10], etc. The LaSalle theorem was developed in [4], and the Lyapunov method was applied by many authors to deal with stochastic property (e.g., see [5]

and [6]). It should be pointed out that the linear growth condition was required in [1, 3, 4, 6]. However, the linear growth condition is sometimes too strong to be satisfied in our real lives. Therefore, it is interesting and challenging to study the stability of stochastic systems when they do not satisfy the linear growth condition. In this paper, we consider a class of nonlinear differential delay equations (DDEs) with Poisson jump:

$$\begin{aligned} dx(t) = & f(x(t), x(t - \tau(t)), t)dt \\ & + g(x(t), x(t - \tau(t)), t)dw(t) \\ & + \int_{\mathcal{Z}} h(x(t), x(t - \tau(t)), t, v)N(dt, dv), t \geq 0 \end{aligned} \quad (1)$$

Under the nonlinear growth condition, we prove that there exists a unique global solution of DDEs with poisson jumps (1) by using Itô formula, Grownwall’s inequality and nonnegative semi-martingales convergence theorem. Furthermore, we establish the stability in mean square of system (1). Moreover, we prove the existence and the stability of the global solution without using the uniform continuous.

The rest of the paper is organized as follows: In Section 2, we give some necessary notations and lemmas. In Section 3, we discuss the existence-and-uniqueness of the global solution as well as the mean square exponential stability.

2 RESEARCH METHODS

In this work, we introduce the local Lipschitz and new nonlinear growth conditions on coefficients to prove the existence of a unique global solution. Then by applying Lyapunov function method, we prove that the stochastic system under consideration has a unique global solution and investigate the exponential stability in the mean square of the solution.

3 PRELIMINARIES

In this paper, unless otherwise specified, we will employ the following notations. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition (i.e., it is increasing and right continuous while contains all \mathbf{P} -null sets). Let $|x|$ denote the Euclidean norm of a vector in \mathbb{R}^n . If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_m(t))^T$ ($t \geq 0$) be an m -dimensional Brownian motion defined on the probability space. $a \vee b$ denotes the maximum value between a and b , while $a \wedge b$ denotes the minimum value. $C([-\tau, 0]; \mathbb{R}^n)$ is the family of continuous functions from $[-\tau, 0]$ to \mathbb{R}^n , and $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel algebra in \mathbb{R}^n .

Let $\bar{p} = \{\bar{p}(t), t \geq 0\}$ be a stationary \mathcal{F}_t -adapted and \mathbb{R}^n -valued Poisson point process. For $A \in \mathcal{B}(\mathbb{R}^n - \{0\})$, here $0 \notin$ the closure of A , we define the Poisson counting measure N associated with \bar{p}

by

$$N((0, t] \times A) := \#\{0 < s \leq t, \bar{p}(s) \in A\} \\ = \sum_{t_0 < s \leq t} I_A(\bar{p}(s)),$$

where $\#$ denotes the cardinality of set $\{\cdot\}$. For simplicity, we denote $N(t, A) := N((0, t] \times A)$. It is known that there exists a σ -finite measure π such that

$$E[N(t, A)] = \pi(A)t, \quad P(N(t, A) = n) \\ = \frac{\exp(-t\pi(A))(\pi(A)t)^n}{n!}.$$

This measure π is called the Lévy measure. Denote by $N(t, z)$ a \mathcal{F}_t -adapted Poisson random martingale measure $\tilde{N}(t, A)$ satisfies

$$N(t, A) = \tilde{N}(t, A) + \hat{N}(t, A), t > 0.$$

Here $\tilde{N}(t, A)$ denotes the compensated Poisson random measure and $\hat{N}(t, A) = \pi(A)t$ denotes the compensator.

In this paper, we assume that Poisson random measure N is independent of Brownian motion ω . For $Z \in \mathcal{B}(\mathbb{R}^n - \{0\})$, consider the following nonlinear SDDEs with Poisson jump:

$$dx(t) = f(x(t), x(t - \tau(t)), t)dt \\ + g(x(t), x(t - \tau(t)), t)d\omega(t) \\ + \int_Z h(x(t), x(t - \tau(t)), t, v)N(dt, dv), \tag{2}$$

where initial value $x(0) = x_0 = \xi \in C([-\tau, 0]; \mathbb{R}^n)$, $\tau(\cdot) : [0, +\infty) \rightarrow [0, \tau]$, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times m}$ and $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times Z \rightarrow \mathbb{R}^n$. For the purpose of the stability, we assume that $f(0, 0, t) = 0, g(0, 0, t) = 0$.

Definition 3.1. The trivial solution of (2) is said to be asymptotically stable in the p th moment if

$$\lim_{t \rightarrow \infty} \mathbf{E}|x(t)|^p = 0$$

for all $x_0 \in \mathbb{R}^n$. We remark that asymptotically stable in the p -th moment is often called asymptotically stable in the mean square when $p = 2$.

The aim of this section is to discuss the existence and uniqueness of solution to system (2). To this end, we first impose some necessary conditions on two functions f and g .

Assumption 3.2. (The Local Lipschitz Condition) For every integer $k \geq 1$, there is $C_k > 0$ such that

$$\begin{aligned} &|f(x_1, y_1, t) - f(x_2, y_2, t)|^2 \\ &\vee |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 \\ &\leq C_k \left(|x_1 - x_2|^2 + |y_1 - y_2|^2 \right), \\ &\int_Z |h(x_1, y_1, t, v) - h(x_2, y_2, t, v)|^2 \pi(dv) \\ &\leq C_k \left(|x_1 - x_2|^2 + |y_1 - y_2|^2 \right), \end{aligned} \tag{3}$$

for all $(x_j, y_j, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, $|x_j| \vee |y_j| \leq k$ ($j = 1, 2$).

Obviously, it follows from Assumption 2.1 that there exists a unique maximal local solution on $[-\tau, \tau_e)$ to system (2), where τ_e is the explosion time. For the convenience of the reader, we state this fact as the following lemma.

Lemma 3.3. *Let Assumption 3.2 hold. Then for any initial value ξ , system (2) has a unique maximal local solution on $-\tau < t < \tau_e$.*

To ensure that the existence and uniqueness as well as stability of the solution to system (2), W. Zhou [7] used the local Lipschitz condition and linear growth condition. However, the linear growth condition is very restrict and many real models do not satisfy it. In this paper, we will introduce a new general nonlinear condition to replace the linear growth condition, and under it we can also prove the existence and uniqueness as well as stability of the solution to the system (2). Now, let us introduce the new general nonlinear growth condition as follows.

Assumption 3.4. For all $x, y \in \mathbb{R}^n$, $t \geq 0$, there exist four nonnegative constants a_1, a_2, a_3 and a_4 , such that

$$\begin{aligned} &x^T f(x, y, t) + \frac{1}{2} |g(x, y, t)|^2 \\ &\leq a_1 |x|^2 - a_2 |x|^{r_1+2} + a_3 |x|^{r_1+2} + a_4 |y|^{r_2+2}, \end{aligned} \tag{4}$$

where $r_1 > 0$ and $r_2 > 0$ are constants.

Assumption 3.5. For all $x, y \in \mathbb{R}^n$ and $t \geq 0$, there exist three nonnegative constants b_1, b_2, b_3 and a function $\bar{h}(v)$, such that

$$\begin{aligned} &|x + h(x, y, t, v)|^2 \\ &\leq \bar{h}(v) (b_1 |x|^2 + b_2 |x|^{r_3+2} + b_3 |y|^{r_3+2}), \end{aligned} \tag{5}$$

where $r_3 > 0$ is a constant and the function $\bar{h}(v)$ satisfies $C_{\bar{h}} = \int_Z \bar{h}(v) \pi(dv) < \infty$.

Assumption 3.6. $\tau(\cdot)$ is continuously differentiable and there exists a constant u such that

$$\tau'(t) \leq u < 1.$$

Denote by $C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ the family of all non-negative continuous functions $V(t, x)$, which are continuously twice differentiable in x and once in t defined on $\mathbb{R}_+ \times \mathbb{R}^n$. Given $V \in C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$, we define an operator $\mathcal{L}V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x, y, t) &= V_t(x, t) + V_x(x, t)f(x, y, t) \\ &+ \frac{1}{2} \text{tr} [g^T(x, y, t)V_{xx}(x, t)g(x, y, t)] \\ &+ \int_Z [V(x + h(x, y, t, v), t) - V(x, t)]\pi(dv), \end{aligned} \tag{6}$$

where $V_t(x, t) = \frac{\partial V(x, t)}{\partial t}$, $V_x(x, t) = \left(\frac{\partial V(x, t)}{\partial x_1}, \dots, \frac{\partial V(x, t)}{\partial x_n} \right)$, $V_{xx}(x, t) = \left(\frac{\partial^2 V(x, t)}{\partial x_i \partial x_j} \right)_{n \times n}$.

Lemma 3.7. (The semi-martingale convergence theorem [8]). *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0)$ a.s.. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s., and let ξ be a nonnegative F_0 -measurable random variable such that $E\xi < \infty$. Define $X(t) = \xi + A(t) - U(t) + M(t)$ for $t \geq 0$. If $X(t)$ is nonnegative, then*

$$\begin{aligned} &\left\{ \lim_{t \rightarrow \infty} A(t) < \infty \right\} \subset \\ &\left\{ \lim_{t \rightarrow \infty} X(t) < \infty \right\} \cap \left\{ \lim_{t \rightarrow \infty} U(t) < \infty \right\} \text{ a.s.,} \end{aligned}$$

where $C \subset D$ a.s., means $\mathbf{P}(C \cap D^c) = 0$. In particular, if $\lim_{t \rightarrow \infty} A(t) < \infty$ a.s., then with probability one, $\lim_{t \rightarrow \infty} X(t) < \infty, \lim_{t \rightarrow \infty} U(t) < \infty, -\infty < M(t) < +\infty$.

4 MAIN RESULTS

In this section, we assume that the functions f and g satisfy the local Lipschitz condition and new general growth condition. Under these conditions, we will first prove that there exists a unique global solution to system (2). Furthermore, we will discuss the mean square exponential stability of the solution.

Theorem 4.1. *Suppose that Assumptions 3.2, 3.4, 3.5 and 3.6 hold. If the following conditions hold:*

$$r_1 \geq r_2 \vee r_3; -2a_1 + b_1 C_{\bar{h}} - \pi(Z) < 0$$

and

$$a_2 > a_3 + \left(\frac{u}{1-u} + 1\right) a_4 + \frac{1}{2} \left(b_2 + \frac{1}{1-u} b_3\right) C_{\bar{h}}. \quad (7)$$

Then for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$, there exists a global unique solution $x(t)$ to equation (2) on $t \in [-\tau, \infty)$. Moreover, the trivial solution of (2) is mean square asymptotically exponential stability.

Proof. Since the functions f and g satisfy the local Lipschitz condition, we see that for any given initial value $x(0) = x_0 = \xi \in \mathbb{R}^n$, there exists a unique maximum local strong solution on $-\tau < t < \tau_e$, where τ_e is the explosion time. To prove that this solution is global, we just need to show that $\tau_e = \infty$ a.s.. Let $k_0 > 0$ be sufficiently large such that $k_0 > |x_0|$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf \{t \in [0, \tau_e) : |x(t)| \geq k\}, \quad k \in \mathbb{N}.$$

By the definition of the stopping time τ_k , it is obvious that τ_k is increasing with k , so $\tau_k \rightarrow \tau_\infty \leq \tau_e (k \rightarrow \infty)$ a.s., where $\inf \emptyset = \infty$ (as usual, $\emptyset =$ the empty set). If we can prove $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s., which implies that $x(t)$ is the unique global solution. Hence, we only need to prove that $\mathbf{P}(\tau_k \leq t) \rightarrow 0 (k \rightarrow \infty, t > 0)$. We define $V(x(t), t) = |x(t)|^2$, it is easy to get $EV(x(t \wedge \tau_k), t \wedge \tau_k) \geq \mathbf{P}(\tau_k \leq t) V(x(\tau_k), \tau_k)$. That is to say, we only need to prove that $EV(x(t \wedge \tau_k), t \wedge \tau_k) < +\infty$ since $V(x(\tau_k), \tau_k) = |x(\tau_k)|^2 = k^2 \rightarrow \infty (k \rightarrow \infty)$. By the Itô formula, we obtain

$$EV(x(t \wedge \tau_k), t \wedge \tau_k) = V(x(0), 0) + E \int_0^{t \wedge \tau_k} \mathcal{L}V(x(s), y(s), s) ds, \quad (8)$$

where

$$\begin{aligned} \mathcal{L}V(x(t), y(t), t) &= V_t(x(t), t) + V_x(x(t), t)f(x(t), y(t), t) \\ &+ \frac{1}{2} \text{tr} [g^T(x(t), y(t), t)V_{xx}(x(t), t)g(x(t), y(t), t)] \\ &+ \int_Z [V(x+h(x, y, t, v), t) - V(x(t), t)]\pi(dv) \\ &\leq 2 \left[x^T f(x(t), y(t), t) + \frac{1}{2} |g(x(t), y(t), t)|^2 \right] \\ &+ \int_Z [|x+h(x, y, t, v)|^2 - |x(t)|^2] \pi(dv). \quad (9) \end{aligned}$$

According to Assumptions 3.2 and 3.4, we have

$$\begin{aligned} \mathcal{L}V(x(t), y(t), t) &\leq [a_1|x(t)|^2 - a_2|x(t)|^{\gamma_1+2} + a_3|x(t)|^{\gamma_1+2} \\ &+ a_4|y(t)|^{\gamma_2+2}] \\ &+ \int_Z [|x+h(x, y, t, v)|^2 - |x(t)|^2] \pi(dv). \quad (10) \end{aligned}$$

By the inequality $|\alpha + \beta + \gamma| \leq (|\alpha| + |\beta| + |\gamma|)$, we obtain

$$\begin{aligned} |x+h(x, y, t, v)|^2 &= (|x+h(x, y, t, v)|^2) \\ &\leq (\bar{h}(v) (b_1|x(t)|^2 + b_2|x(t)|^{r_3+2} + b_3|y(t)|^{r_3+2})) \\ &= \bar{h}(v) (b_1|x(t)|^2 + b_2|x(t)|^{r_3+2} + b_3|y(t)|^{r_3+2}) \\ &\leq \bar{h}(v) (b_1|x(t)|^2 + b_2|x(t)|^{r_3+2} \\ &+ b_3|y(t)|^{r_3+2}). \quad (11) \end{aligned}$$

Thus, we get

$$\begin{aligned} &\int_Z [|x+h(x, y, t, v)|^2 - |x(t)|^2] \pi(dv) \\ &\leq b_1 C_{\bar{h}} |x(t)|^2 + b_2 C_{\bar{h}} |x(t)|^{r_3+2} \\ &+ b_3 C_{\bar{h}} |y(t)|^{r_3+2} - \pi(Z) |x(t)|^2, \quad (12) \end{aligned}$$

where $C_{\bar{h}} = \int_Z 2(\bar{h}(v))\pi(dv)$.

Substituting (11) and (12) into (10) yields

$$\begin{aligned} \mathcal{L}V(x(t), y(t), t) &\leq 2a_1|x(t)|^2 - 2a_2|x(t)|^{r_1+2} \\ &+ 2a_3|x(t)|^{r_2+2} + 2a_4|y(t)|^{r_2+2} \\ &+ b_1 C_{\bar{h}} |x(t)|^2 + b_2 C_{\bar{h}} |x(t)|^{r_3+2} \\ &+ b_3 C_{\bar{h}} |y(t)|^{r_3+2} - \pi(Z) |x(t)|^2. \quad (13) \end{aligned}$$

By utilizing the Young inequality $\alpha^2\beta^2 \leq \frac{1}{2}\alpha^4 + \frac{1}{2}\beta^4 (\alpha, \beta \geq 0)$, we have

$$|y(t)|^{r_2+2} \leq |x(t)|^{2+r_2} + (|y(t)|^{2+r_2} - |x(t)|^{2+r_2}). \quad (14)$$

Similarly, we get

$$\begin{aligned} &b_2 C_{\bar{h}} |x(t)|^{r_3+2} + b_3 C_{\bar{h}} |y(t)|^{r_3+2} \\ &= (b_2 C_{\bar{h}} + b_3 C_{\bar{h}}) |x(t)|^{r_3+2} \\ &+ b_3 C_{\bar{h}} (|y(t)|^{r_3+2} - |x(t)|^{r_3+2}). \quad (15) \end{aligned}$$

Substituting (14)-(15) into (13) yields

$$\begin{aligned} \mathcal{L}V(x(t), y(t), t) &\leq (2a_1 + b_1 C_{\bar{h}} - \pi(Z)) |x(t)|^2 \\ &+ a_4 (|y(t)|^{2+r_2} - |x(t)|^{2+r_2}) \\ &+ b_3 C_{\bar{h}} (|y(t)|^{r_3+2} - |x(t)|^{r_3+2}) - 2a_2|x(t)|^{r_1+2} \\ &+ (2a_3 + 2a_4) |x(t)|^{r_2+2} \\ &+ (b_2 + b_3) C_{\bar{h}} |x(t)|^{r_3+2}. \quad (16) \end{aligned}$$

Substituting this fact into (8) yields

$$\begin{aligned}
 EV(x(t \wedge \tau_k), t \wedge \tau_k) &\leq V(x(0), 0) \\
 &+ (2a_1 + b_1 C_{\bar{h}} - \pi(Z)) E \int_0^{t \wedge \tau_k} |x(s)|^2 ds \\
 &+ 2a_4 E \int_0^{t \wedge \tau_k} (|y(s)|^{2+r_2} - |x(s)|^{2+r_2}) ds \\
 &+ b_3 2C_{\bar{h}2} E \int_0^{t \wedge \tau_k} (|y(s)|^{r_3+2} - |x(s)|^{r_3+2}) ds \\
 &- 2a_2 E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_1} ds \\
 &+ (2a_3 + 2a_4) E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_2} ds \\
 &+ (b_2 + b_3) C_{\bar{h}} \int_0^{t \wedge \tau_k} E|x(t)|^{r_3+2} ds. \tag{17}
 \end{aligned}$$

According to the integral property, we have

$$\begin{aligned}
 \int_0^{t \wedge \tau_k} |y(s)|^{2+r_2} ds &\leq \frac{1}{1-u} \int_{-\tau(0)}^{t \wedge \tau_k - \tau(t)} |x(s)|^{2+r_2} ds \\
 &\leq \frac{1}{1-u} \int_{-\tau}^{t \wedge \tau_k} |x(s)|^{2+r_2} ds,
 \end{aligned}$$

then we can easily get

$$\begin{aligned}
 &\int_0^{t \wedge \tau_k} (|y(s)|^{2+r_2} - |x(s)|^{2+r_2}) ds \\
 &\leq \frac{1}{1-u} \int_{-\tau}^{t \wedge \tau_k} |x(s)|^{2+r_2} ds \\
 &\quad - \int_0^{t \wedge \tau_k} |x(s)|^{2+r_2} ds \\
 &\leq \frac{1}{1-u} \int_{-\tau}^0 |x(s)|^{2+r_2} ds \\
 &\quad + \frac{u}{1-u} \int_0^{t \wedge \tau_k} |x(s)|^{2+r_2} ds.
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 &\int_0^{t \wedge \tau_k} (|y(s)|^{2+r_3} - |x(s)|^{2+r_3}) ds \\
 &\leq \frac{1}{1-u} \int_{-\tau}^0 |x(s)|^{2+r_3} ds \\
 &\quad + \frac{u}{1-u} \int_0^{t \wedge \tau_k} |x(s)|^{2+r_3} ds.
 \end{aligned}$$

Substituting the above inequalities into (17) yields

$$\begin{aligned}
 EV(x(t \wedge \tau_k), t \wedge \tau_k) &\leq V(x(0), 0) \\
 &+ (2a_1 + b_1 C_{\bar{h}} - \pi(z)) E \int_0^{t \wedge \tau_k} |x(s)|^2 ds \\
 &+ \frac{2}{1-u} a_4 E \int_{-\tau}^0 |x(s)|^{r_2+2} ds \\
 &+ \frac{1}{1-u} b_3 C_{\bar{h}} E \int_{-\tau}^0 |x(s)|^{r_3+2} ds \\
 &- 2a_2 E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_1} ds \\
 &+ \left(2a_3 + \left(1 + \frac{u}{1-u} \right) 2a_4 \right) \\
 &\quad \times E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_2} ds \\
 &+ \left(b_2 + \frac{1}{1-u} b_3 \right) C_{\bar{h}} E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_3} ds \\
 &= V(x(0), 0) \\
 &+ (2a_1 + b_1 C_{\bar{h}} - \pi(z)) E \int_0^{t \wedge \tau_k} |x(s)|^2 ds \\
 &+ \frac{2}{1-u} a_4 E \int_{-\tau}^0 |x(s)|^{r_2+2} ds \\
 &+ \frac{1}{1-u} b_3 C_{\bar{h}} E \int_{-\tau}^0 |x(s)|^{r_3+2} ds + I(x(t), t), \tag{18}
 \end{aligned}$$

where

$$\begin{aligned}
 I(x(t), t) &= -2a_2 E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_1} ds \\
 &+ \left(2a_3 + \left(1 + \frac{u}{1-u} \right) 2a_4 \right) E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_2} ds \\
 &+ \left(b_2 + \frac{1}{1-u} b_3 \right) C_{\bar{h}} E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_3} ds.
 \end{aligned}$$

Recalling that $r_1 \geq r_2 \vee r_3$ and $a_2 \geq a_3 + \left(1 + \frac{u}{1-u} \right) a_4 + \frac{1}{2} \left(b_2 + \frac{1}{1-u} b_3 \right) C_{\bar{h}}$, the boundness of the polynomial function implies that there exists H_0 such that $I(x(t), t) \leq H_0$. Then we can deduce from (18) that

$$\begin{aligned}
 EV(x(t \wedge \tau_k), t \wedge \tau_k) &\leq E\|\xi\|^2 + H_0 t \\
 &+ (2a_1 + b_1 C_{\bar{h}} - \pi(z)) E \int_0^t |x(t \wedge \tau_k)|^2 ds \\
 &+ \frac{2}{(1-u)} a_4 E \int_{-\tau}^0 |x(s)|^{r_2+2} ds \\
 &+ \frac{1}{1-u} b_3 C_{\bar{h}} E \int_{-\tau}^0 |x(s)|^{r_3+2} ds. \tag{19}
 \end{aligned}$$

Applying the Gronwall inequality to (19), we get

$$EV(x(t \wedge \tau_k)) \leq H_t e^{(-2a_1 + b_1 C_{\bar{h}} - \pi(Z))t},$$

where

$$\begin{aligned}
 H_t &= H_0 t + E\|\xi\|^2 + \frac{2}{(1-u)} a_4 \tau E\|\xi\|^{r_2+2} \\
 &\quad + \frac{1}{1-u} b_3 C_{\bar{h}} \tau E\|\xi\|^{r_3+2} \\
 H_0 t &= \sup_{x \geq 0} \left\{ -2a_2 E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_1} ds \right. \\
 &\quad + \left(2a_3 + \left(1 + \frac{u}{1-u} \right) 2a_4 \right) E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_2} ds \\
 &\quad \left. + \left(b_2 + \frac{1}{1-u} b_3 \right) C_{\bar{h}} E \int_0^{t \wedge \tau_k} |x(s)|^{2+r_3} ds \right\}. \tag{20}
 \end{aligned}$$

Obviously, H_t is dependent of t , but independent of k . Letting $k \rightarrow \infty$, we obtain

$$E|x(t)|^2 \leq H_t e^{(-2a_1 + b_1 C_{\bar{h}} - \pi(Z))t}.$$

Since $t > 0$ is arbitrary, which implies that for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$, there exists a unique global solution $x(t)$ on $[-\tau, \infty)$ and the trivial solution of (2) is mean square asymptotically exponential stability.

5 CONCLUSION

In this paper, we have introduced a new general nonlinear growth condition and under it we have established the existence-uniqueness and stability of the global solution.

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