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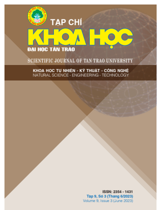

ON THE STABILITY OF SOLUTIONS FOR THE SEMI-AFFINE VARIATIONAL INEQUALITIES IN HILBERT SPACES

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Abstract:

This paper investigates the stability of solutions for the class of semi-affine variational inequalities in real Hilbert spaces whose feasible set does not depend on parametric. As an application, we obtain the stability of solutions for the parametric quadratically constrained quadratic programming problems in real Hilbert spaces.

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VỀ TÍNH ỔN ĐỊNH NGHIỆM CỦA BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN NỬA AFFINE TRONG KHÔNG GIAN HILBERT

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Từ khóa:
Nửa affine, Bất đẳng thức biến phân, không gian Hilbert, tính ổn định nghiệm, bài toán nhiễu.

Tóm tắt:

Bài báo này quan tâm đến tính ổn định nghiệm của bài toán bất đẳng thức biến phân nửa Affine trong không gian Hilbert thực trong đó tập ràng buộc không phụ thuộc vào tham số. Như một ứng dụng, chúng ta đạt được tính ổn định nghiệm của bài toán quy hoạch toàn phương trong không gian Hilbert thực.

1 INTRODUCTION

Let us assume that $\mathcal{H}$ is a real Hilbert space which is endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ and let K be a closed convex set in $\mathcal{H}$. The *semi-affine variational inequalities* is the problem of find $x \in K$ such that

$$\langle Tx + c, y - x \rangle \geq 0 \ \ \forall y \in K, \tag{1}$$

where T (with the adjoint T^*) be a bounded linear operators on $\mathcal{H}$, $c \in \mathcal{H}$.

The variational inequality theory which is mainly due to Stampacchia (see [8]) provides very powerful techniques for studying problems arising in mechanics, optimization, transportation, economics equilibrium, contact problems in elasticity, and other branches of mathematics. Problems *semi-affine variational inequalities* including *linear complementarity problems*, give a suitable format for many problems arising in economics, mathematical physics, operations research, mathematical programming and have been extensively studied in literature both in finite- or infinite-dimensional spaces (see, e.g, [8, 5, 10, 9] and references therein).

Since problems of the form (1) is a subclass of variational inequality problem, the stability of variational inequality problem can be applied to semi-affine variational inequality problems. However, the special structure of semi-affine variational inequality allows one to have deeper and sharper results on the stability properties of the form (1).

This paper investigates the conditions for particular points to be solution and stability of the solutions for a class of semi-affine variational inequalities whose constraint set is defined by finitely many convex linear quadratic inequalities in Hilbert spaces. As an application, we obtain stability of solutions for the parametric quadratically constrained quadratic programming problems in Hilbert spaces.

The remainder of the paper is organized as follows. Some preliminaries are given in Section 2. In the next section, we propose conditions for a feasible point to be a solution of a class of semi-affine variational inequalities problems. A stability result of the solution for problem is also investigated in this section. Finally, in Sect. 4, we obtain the stability of solutions for the parametric quadratically constrained quadratic programming problems in Hilbert spaces by applying the results in the previous section.

2 NOTATIONS AND PRELIMINARY RESULTS

In this paper, the set K is defined by the finitely many convex quadratic constraints of the form

$$K = \{x \in \mathbb{R}^n \mid g_i(x) := \frac{1}{2}\langle x, T_i x\rangle + \langle c_i, x\rangle + \alpha_i \leq 0\},$$

where T_i is a positive semidefinite continuous linear self-adjoint operator on $\mathcal{H}$, $c_i \in \mathcal{H}$, and α_i are real numbers, $i = 1, 2, \ldots, m$.

In this section we recall some notations and known results which will be used in our analysis. For details, we refer to [1].

In this paper, $\operatorname{dist}(x, S) = \inf\limits_{y \in S} \|x - y\|$ stands for the distance from the point $x \in \mathcal{H}$ to set $S \subset \mathcal{H}$. The norm of a continuous linear operator $Q : \mathcal{H} \to \mathcal{H}$ shall be defined $\|Q\| = \sup\Big\{\dfrac{\|Qx\|}{\|x\|} \mid x \in \mathcal{H}, x \neq 0\Big\}$.

The notation $r(h) = O(h)$ means that $\dfrac{r(h)}{\|h\|}$ is bounded for all h in neighborhood of $0 \in X$, where X is Banach or locally convex topological vector spaces and $r(t) = o(t)$ mean that $\dfrac{r(t)}{t} \to 0$ as $t \to 0$,.

Definition 2.1. [See, e.g., [1, p.45]] Let $x \in K$ and denote by $I(x) = \{i \in \{1, 2, \ldots, m\} \mid g_i(x) = 0\}$ the set of inequality constraints active at x, as well as by

$$T_K(x) = \{h \in \mathcal{H} \mid \ \operatorname{dist}(x + th, K) = o(t), t \geqslant 0\},$$

the *tangent cone* of K at x.

Definition 2.2. [See, e.g., [1, p.71]] (Mangasarian-Fromovitz constraint qualification) *The feasible point $\bar{x}$ is called regular if*

$$\exists h \in \mathcal{H} : \langle T_i \bar{x} + c_i, h \rangle < 0, \forall i \in I(\bar{x}). \tag{2}$$

Remark 2.1. Note that if $\bar{x} \in K$ is regular, then $T_K(\bar{x})$ is formulated as follows (see [1, Example 3.39])

$$T_K(\bar{x}) = \{h \in \mathcal{H} \mid \langle Q_i \bar{x} + c_i, h \rangle \leqslant 0, \forall i \in I(\bar{x})\}.$$

We will need the following lemma, which is an extension of a Hoffman estimate for the distance to the set of solutions to a system of linear inequalities.

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We will need the following lemma, which is an extension of a Hoffman estimate for the distance to the set of solutions to a system of linear inequalities.

Lemma 2.1 (see [7, Theorem 3]). *Let $\mathcal{H}$ be a Hilbert space. Let $x_i^* \in \mathcal{H}, i = 1, 2, \ldots, m$, be given, and consider the set*

$$S = \{x \in \mathcal{H} \mid \langle x_i^*, x\rangle \leqslant 0,\ i = 1, 2, \ldots, m\}.$$

Then there exists a constant $k > 0$ such that for any $x \in \mathcal{H}$,

$$\mathrm{dist}(x, S) \leqslant k\Big(\sum_{i=1}^{m} \langle x_i^*, x\rangle_+\Big),$$

where $[a]_+ := \max\{a, 0\}$.

3 MAIN RESULTS

Let $\mathcal{L}(\mathcal{H})$ be the space of continuous linear operators from $\mathcal{H}$ into $\mathcal{H}$ equipped with the operator norm induced by the vector norm in $\mathcal{H}$ and also denoted by $\|\cdot\|$. The norm in the product space $X_1 \times \ldots \times X_k$ of the normed spaces $X_1, \ldots, X_k$ is defined by $\|(x_1, \ldots, x_k)\| = \max\{\|x_1\|, \ldots, \|x_k\|\}$.
Consider the perturbed variational inequality: find $x \in K$ such that

$$\langle Tx + c, y - x\rangle \geq 0 \ \ \forall y \in K \qquad (\mathrm{sAVI}\omega)$$

depending on the parameter vector $\omega = (T, c) \in \Omega = \mathcal{L}(\mathcal{H}) \times \mathcal{H}$. For a given point ω_0, in the parameter space Ω, we view the corresponding problem $(\mathrm{sAVI}\omega_0)$ as an unperturbed problem. We denote by $S(\omega)$ the set of solutions of $(\mathrm{sAVI}\omega)$. For a given value ω_0 of the parameter vector we assume that $(\mathrm{GAVI}_{\omega_0})$ coincides with the unperturbed problem.
The following result gives a set of necessary and sufficient conditions for $\bar{x}$ to be a solution of $(\mathrm{sAVI}\omega_0)$. The proof of this theorem is similar to the proof of Theorem 2.1 in [4]. However, for the sake of completeness, we give the complete proof here.

Theorem 3.1.

(i) *Suppose that $\bar{x}$ is a solution of* $(\mathrm{sAVI}\omega_0)$*, and that there exists $x^0 \in \mathcal{H}$ be such that $g_i(x^0) < 0$ for all $i = 1, 2, \ldots, m$ (Slater condition). Then, there exists $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ such that*

$$\begin{cases} T\bar{x} + c + \sum\limits_{i=1}^{m} \lambda_i(T_i\bar{x} + c_i) = 0 \\ \frac{1}{2}\langle \bar{x}, T_i\bar{x}\rangle + \langle c_i, \bar{x}\rangle + \alpha_i \leq 0, \\ \lambda_i(\frac{1}{2}\langle \bar{x}, T_i\bar{x}\rangle + \langle c_i, \bar{x}\rangle + \alpha_i) = 0, \\ \lambda_i \geq 0,\ i = 1, \ldots, m. \end{cases} \qquad (3)$$

(ii) *If there exist $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ and $\bar{x} \in \mathcal{H}$ such that the* (3) *is satisfied, then $\bar{x}$ is a solution of* $(\mathrm{sAVI}\omega_0)$.

Proof. (i) Suppose that $\bar{x}$ is a solution of $(\mathrm{sAVI}\omega_0)$. Then,

$$\langle T\bar{x} + c, y - \bar{x}\rangle \geq 0 \quad \forall y \in K,$$

which is equivalent to the following one

$$\langle T\bar{x} + c, \bar{x}\rangle \leq \langle T\bar{x} + c, y\rangle \quad \forall y \in K.$$

Hence $\bar{x}$ is an optimal solution of the optimization problem

$$\min_{y\in K} \langle T\bar{x} + c, y\rangle. \qquad (4)$$

We have

$$\langle T_i\bar{x} + c_i, \bar{x} - x^0\rangle = \lim_{t\downarrow 0} \frac{g_i(\bar{x} + t(x^0 - \bar{x})) - g_i(\bar{x})}{t}. \qquad (5)$$

Since T_i are positive semi-definite continuous linear self-adjoint operators, $c_i \in \mathcal{H}$, it follows that g_i are continuous and convex. By convexity of g_i, we have

$$\begin{aligned} g_i(\bar{x} + t(x^0 - \bar{x})) &= g_i((1-t)\bar{x} + tx^0) \\ &\leq (1-t)g_i(\bar{x}) + tg_i(x^0) \ \ \forall t \in (0, 1). \end{aligned} \qquad (6)$$

Combining (5) with (6) we obtain

$$\langle T_i\bar{x} + c_i, \bar{x} - x^0\rangle \leq g_i(x^0) - g_i(\bar{x})$$

Put $\bar{h} = \bar{x} - x^0$. It follows from the above inequality that

$$\langle T_i\bar{x} + c_i, \bar{h}\rangle < 0, \ \ \forall i \in I(\bar{x}),$$

where $I(x) = \{i \in \{1, 2, \ldots, m\} \mid g_i(x) = 0\}$. Hence $T_K(\bar{x})$, the *tangent cone* of K at $\bar{x}$, is formulated as follows (see [1, Example 3.39])

$$T_K(\bar{x}) = \{h \in \mathcal{H} \mid \langle T_i\bar{x} + c_i, h\rangle \leq 0, \forall i \in I(\bar{x})\}.$$

Since $\bar{x}$ is a solution of (4) and by Lemma 3.7 in [1], it follows that $h = 0$ is an optimal solution of the problem

$$\min_{h\in\mathcal{H}} \langle T\bar{x} + c, h\rangle \text{ subjects to } \langle T_i\bar{x} + c_i, h\rangle \leq 0, i \in I(\bar{x}). \qquad (7)$$

The (7) is a linear programming problem with a finite (equal zero) optimal value. By Hoffman's lemma (see, for instance, [7, Theorem 3]), we have that the set of optimal solutions of the dual problem of (7)

$$\max_{\lambda_i \geq 0} 0 \quad \text{subjects to} \quad T\bar{x} + c + \sum_{i\in I(\bar{x})} \lambda_i(T_i\bar{x} + c_i) = 0 \qquad (8)$$

is nonempty.

Put $\lambda_i = 0$ for all $i \in I \backslash I(\bar{x})$, and $\lambda = (\lambda_1, \ldots, \lambda_m)$. From (8) we obtain the first equality in (3). Since $\bar{x} \in K$ and $\lambda_i(\frac{1}{2}\langle \bar{x}, T_i\bar{x}\rangle + \langle c_i, \bar{x}\rangle + \alpha_i) = 0$ for each $i \in I$, the other conditions in (3) are satisfied too.

(ii) Suppose that the (3) is satisfied. Then, for every $y \in K$ we have $g_i(y) \leq 0$ and

$$\begin{aligned}-\langle T\bar{x} + c, y - \bar{x}\rangle &= \sum_{i=1}^{m} \lambda_i \langle T_i\bar{x} + c_i, y - \bar{x}\rangle \\ &\leq \sum_{i=1}^{m} \lambda_i [g_i(y) - g_i(\bar{x})] = \sum_{i=1}^{m} \lambda_i g_i(y).\end{aligned}$$

From the above it follows that $\bar{x}$ is a solution of (sAVIω_0). □

The following example shows that the conclusion of Theorem 3.1 (i) fails if the assumption on Slater condition does not holds.

Example 3.1. Consider the problem (sAVIω_0) where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $Tx = (x_1, 0)$, $c = (0, -1)$ and $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T_1x = (x_1 - x_2, -x_1 + x_2)$.

Let $K = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid g_1(x) \leq 0\}$. It is easy to check that

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}.$$

It is clear that there does not exist a point $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ such that $g(x^0) < 0$. Hence the Slater condition does not holds.

For $\bar{x} = (1, 1) \in \mathbb{R}^2$ we have

$$\begin{aligned}\langle T\bar{x} + c, x - \bar{x}\rangle &= \langle (1, -1), (x_1 - 1, x_1 - 1)\rangle \\ &= x_1 - 1 - x_1 + 1 = 0\end{aligned}$$

for all $x \in K$. Hence $\bar{x} = (1, 1)$ is a solution of (sAVIω_0).

Since $T\bar{x} + c = (1, -1)$ and $T_1\bar{x} = 0$, we see that there exists no $\lambda_1 \geq 0$ such that $T\bar{x}+c+\lambda_1 T_1\bar{x} = 0$. Hence first equality in (3) does not hold.

The main result of this section is stated as follows.

Theorem 3.2. *Consider the problem* (sAVIω)*. Let $\bar{x} \in \mathcal{H}$ and $\lambda \in \mathbb{R}^m$ be such that the system*(3) *is satisfied, and suppose that there exist constants $\alpha > 0$, $\gamma > 0$ such that*

$$\langle h, Th\rangle \geq \alpha\|h\| \quad \forall h \in D_\gamma(\bar{x}), \tag{9}$$

where $D_\gamma(\bar{x}) = \{h \in T_K(\bar{x}) \mid \langle T\bar{x} + c, h\rangle \leq \gamma\|h\|\}$. Then $\bar{x} \in S(\omega_0)$ and for all $x(\omega) \in S(\omega)$ in a neighborhood of $\bar{x}$, we have that

$$\|x(\omega) - \bar{x}\| = O(\|\omega - \omega_0\|).$$

Proof. Since there exists $\lambda \in \mathbb{R}^m$ and $\bar{x} \in \mathcal{H}$ such that the system(3) holds, it follows from Theorem 3.1(ii) that $\bar{x} \in S(\omega_0)$.

Let $\omega_k = (T_k, c_k) \to \omega_0 = (T, c)$, and $x_k \in \mathrm{Sol}(\omega_k)$ be such that $x_k \to \bar{x}$ as $k \to \infty$. Set $t_k = \|x_k - \bar{x}\|$ and $h_k = t_k^{-1}(x_k - \bar{x})$, so that $x_k = \bar{x} + t_k h_k$, with $\|h_k\| = 1$. We have

$$\begin{aligned}\langle T_i\bar{x} + c_i, h_k\rangle = \frac{1}{t_k}\{&g_i(x_k) - g_i(\bar{x}) \\ &- \frac{1}{2}\langle x_k - \bar{x}, T_i(x_k - \bar{x})\rangle\} \leqslant 0,\end{aligned}$$

for all $i \in I(\bar{x})$. Set $T_K(\bar{x}) = \{h \in \mathcal{H} \mid \langle T_i\bar{x} + c_i, h\rangle \leqslant 0, \ \ i \in I(\bar{x})\}$. Since $\{h_k\}$ is bounded, it has a weakly convergent subsequence. By passing to a subsequence, we may assume that h_k itself weakly converges to some h. We have

$$\langle T_i\bar{x} + c_i, h\rangle = \lim_{k\to\infty} \langle T_i\bar{x} + c_i, h_k\rangle \leq 0 \ \ \forall i \in I(\bar{x})$$

Hence $h \in T_K(\bar{x})$.

We have prove that $t_k = O(\|\omega_k - \omega_0\|)$. On the contrary, suppose that this is false, i.e., $t_k^{-1}(\omega_k - \omega_0) \to 0$ as $k \to \infty$. Since $\bar{x} \in S(\omega_0)$ and $x_k \in S(\omega_k)$, we have

$$\langle T\bar{x} + c, x_k - \bar{x}\rangle \geq 0 \quad \text{and} \quad \langle T_k x_k + c_k, \bar{x} - x_k\rangle \geq 0$$

Combining these with $x_k - \bar{x} = t_k h_k$ yields

$$\langle T\bar{x} + c, h_k\rangle \geq 0 \quad \text{and} \quad \langle T_k x_k + c_k, -h_k\rangle \geq 0 \tag{10}$$

Letting $k \to \infty$, from (10) we get

$$\langle T\bar{x} + c, h\rangle = 0.$$

Since $\langle T\bar{x} + c, h_k\rangle \geq 0$ and by

$$\langle T\bar{x} + c, h_k\rangle \to \langle T\bar{x} + c, h\rangle = 0 \text{ as } k \to \infty,$$

it follows that there exist sequence γ_k of positive numbers converging to zero such that

$$\langle T\bar{x} + c, h_k\rangle \leq \gamma_k\|h_k\|. \tag{11}$$

Therefore, $h_k \in D_{\gamma_k}(\bar{x})$.

Adding the two inequalities in (10), we obtain

$$\langle T_k x_k + c_k - (T\bar{x} + c), h_k\rangle \leq 0, \tag{12}$$

It is easily verified that (12) is equivalent to

$$\begin{aligned}t_k\langle Th_k, h_k\rangle + t_k\Big\langle \frac{T_k - T}{t_k}\bar{x} + \frac{c_k - c}{t_k}, h_k\Big\rangle& \\ + t_k^2\Big\langle \frac{T_k - T}{t_k} h_k, h_k\Big\rangle \leq 0.&\end{aligned} \tag{13}$$

Since $\dfrac{T_k - T}{t_k} \to 0$ and $\dfrac{c_n - c}{t_k} \to 0$ as $k \to \infty$, it follows that

$$\Big\langle \frac{T_k - T}{t_k}\bar{x} + \frac{c_k - c}{t_k}, h_k \Big\rangle + \Big\langle \frac{T_k - T}{t_n} h_k, h_k \Big\rangle \to 0$$

as $k \to \infty$. Combining this with (13) we have that

$$t_k \langle T h_k, h_k\rangle \le o(t_k).$$

It follows that $\langle Th_k, h_k\rangle < \alpha$ for k large enough. This contradicts (9), and hence the proof is complete. □

It is known that the adjoint operator of T always exists and is bounded linear and unique. Moreover, it is not hard to show that if T^* is an adjoint operator of T, then $M := T + T^*$ is a self adjoint continuous linear operator on $\mathcal{H}$. Recall that if M is a self adjoint bounded linear operator on $\mathcal{H}$, then function $Q : \mathcal{H} \to \mathbb{R}$ defined by $Q(x) = \langle Mx, x\rangle$ is said to be a quadratic form associated with M on $\mathcal{H}$. The quadratic form Q associated with $T + T^*$ is a Legendre form (see, for instance, [6, p.551]) if

(i) it is weakly lower semicontinuous, and

(ii) for any sequance $\{x^k\}$ in $\mathcal{H}$, if $x_k \rightharpoonup x_0$ and $Q(x_k) \to Q(x_0)$, then $x_k \to x_0$.

Theorem 3.3. *Consider the problem* (sAVIω), *where quadratic form associated with* $T + T^*$ *is a Legendre form. Let* $\bar{x} \in \mathcal{H}$ *and* $\lambda \in \mathbb{R}^m$ *be such that the system*(3) *is satisfied and suppose that*

$$\langle h, Th\rangle > 0 \quad \forall h \in D_\gamma(\bar{x})\backslash\{0\}. \tag{14}$$

Then $\bar{x} \in S(\omega_0)$ *and for all* $x(\omega) \in S(\omega)$ *in a neighborhood of* $\bar{x}$, *we have that*

$$\|x(\omega) - \bar{x}\| = O(\|\omega - \omega_0\|).$$

Proof. To prove the theorem, by Theorem 3.2, it suffices to verify that there exists $\alpha > 0$ such that

$$\langle h, Th\rangle \ge \alpha\|h\| \quad \forall h \in D_\gamma(\bar{x}).$$

Suppose that the assertion is fail. Let γ_k be a sequence of positive numbers converging to zero. Then there exists a sequence $h_k \in D_{\gamma_k}(\bar{x})$, $h_k \neq 0$ such that

$$\langle Th_k, h_k\rangle < \frac{1}{k}\|h_k\|^2. \tag{15}$$

Put $v_k := \dfrac{h_k}{\|h_k\|}$, one has $\|v_k\| = 1$. Since $\{v_k\}$ is bounded, it has a weakly convergent subsequence. We may assume that v_k itself weakly converges to some v. Since $h_k \in D(\bar{x})$,

$$\langle T\bar{x} + c, h_k\rangle \le \gamma_k\|h_k\|, \langle T_i\bar{x} + c_i, h_k\rangle \leqslant 0, i \in I(\bar{x}). \tag{16}$$

Multiplying both sides of the inequalities in (16) by $\|h_k\|^{-1}$ and letting $k \to \infty$, we obtain

$$\langle T\bar{x} + c, v\rangle = 0, \langle T_i\bar{x} + c_i, v\rangle \leqslant 0, i \in I(\bar{x}).$$

Hence $v \in D_\gamma(\bar{x})$.

Since the quadratic form Q associated with $T + T^*$ is a Legendre form and by $Q(x) = \langle (T + T^*)x, x\rangle = 2\langle Tx, x\rangle$, it follows that the mapping $x \mapsto \langle Tx, x\rangle$ is weakly lower semicontinuous. Multiplying both sides of the inequalities in (15) by $\|h_k\|^{-2}$ and letting $k \to \infty$, we obtain

$$\begin{aligned}\langle Tv, v\rangle &\le \liminf_{k\to\infty} \Big\langle T\frac{h_k}{\|h_k\|}, \frac{h_k}{\|h_k\|}\Big\rangle \\ &\le \limsup_{k\to\infty} \Big\langle T\frac{h_k}{\|h_k\|}, \frac{h_k}{\|h_k\|}\Big\rangle \le 0.\end{aligned}$$

Combining these with (14) yields

$$v = 0 \ \text{ and } \ \lim_{k\to\infty} Q(v_k) = Q(v).$$

So that $v_k \to v = 0$ by Q is a Legendre form, contrary to the relations $\|v_k\| = 1$. We have thus proved that there exists $\alpha > 0$ such that

$$\langle h, Th\rangle \ge \alpha\|h\| \quad \forall h \in D_\gamma(\bar{x}).$$

The proof is complete. □

Remark 3.1. It is clear that if there exists $\alpha > 0$ such that $\langle h, Th\rangle \ge \alpha\|h\| \quad \forall h \in D_\gamma(\bar{x})$ then $\langle h, Th\rangle > 0 \ \ \forall h \in D_\gamma(\bar{x})\backslash\{0\}$. The converse is not true in general.

Example 3.2. Let ℓ^2 denote the Hilbert space of all square summable real sequence, $\ell^2 = \{x = (x_1, x_2, \ldots) \mid \sum_{n=1}^{\infty} x_n^2 < \infty, x_n \in \mathbb{R}, n = 1, 2, \ldots\}$. Consider the problem (sAVIω) where $\mathcal{H} = \ell^2$, $T : \ell^2 \to \ell^2$ is defined by $Tx = \Big(x_1, \dfrac{x_2}{2^2}, \ldots, \dfrac{x_n}{n^n}, \ldots\Big)$, $c = (0, 0, \ldots, 0, \ldots)$ and the set K is defined

$$K = \big\{x = (x_1, x_2, \ldots) \in \ell^2 \mid \langle c_1, x\rangle - 1 \le 0, \ \big\},$$

with $c_1 = (-1, -\dfrac{1}{2}, \ldots, -\dfrac{1}{n}, \ldots)$.

Taking $\bar{x} = 0$. It is easy to check that $D(\bar{x}) = \mathcal{H}$. Since $Q(x) = \langle Tx, x\rangle = \sum\limits_{n=1}^{\infty} \dfrac{x_n^2}{n^n} > 0$ for all $x \in \ell^2\backslash\{0\}$, $Q(h) > 0 \quad \forall h \in D(\bar{x})\backslash\{0\}$. We have the quadratic form $\langle x, Tx\rangle = \sum\limits_{n=1}^{\infty} \frac{x_n^2}{n^n}$ is not a Legendre form (see, [2, Example 3.3]). Hence $Q(x) = \langle (T+T^*)x, x\rangle = 2\langle Tx, x\rangle$ is not a Legendre form. Therefore, by [6, Theorem 10.1], we deduce that there no exists any α such that $Q(h) \geq \alpha\|h\|^2 \; \forall h \in D(\bar{x})\backslash\{0\}$.

Corollary 3.1. *Consider the problem* (sAVIω). *Let* $\dim\mathcal{H} < \infty$ *and suppose that there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ *and* $\bar{x} \in \mathcal{H}$ *such that the* (3) *holds. Then, if condition* (14) *is satisfied, then* $\bar{x} \in S(\omega_0)$ *and for all* $x(\omega) \in S(\omega)$ *in a neighborhood of* $\bar{x}$*, we have that*

$$\|x(\omega) - \bar{x}\| = O\|\omega - \omega_0\|.$$

Proof. Since the space $\mathcal{H}$ is finite dimensional, any quadratic form $Q : \mathcal{H} \to \mathbb{R}$ is a Legendre form. Hence the assertion follows from Theorem 3.2. $\square$

4 APPLICATIONS TO THE QUADRATIC PROGRAMMING PROBLEMS

In this section we discuss applications of the previous results to the parametric quadratically constrained quadratic programming problems in Hilbert spaces. In what follows, we assume that $T : \mathcal{H} \to \mathcal{H}$ is a continuous linear self-adjoint operator and $c \in \mathcal{H}$. We consider the quadratic programming problem

$$\begin{cases} & \min f(x) := \frac{1}{2}\langle x, Tx\rangle + \langle c, x\rangle \\ \text{s.t.} & x \in K, \end{cases} \tag{QP}$$

depending on the parameter vector $\omega = (T, c) \in \Omega = \mathcal{L}(\mathcal{H}) \times \mathcal{H}$.

Theorem 4.1 (First-order necessary optimality condition)**.** *Suppose that* $\bar{x} \in \mathcal{H}$ *is a local solution of problem* (QP) *and the Slater condition holds. Then, there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ *such that the* (3) *holds.*

Proof. The direct assertion of the above theorem follows from Theorem 3.1 and Theorem 3.2 in [3]. $\square$

Remark 4.1. Suppose that T is a positive semidefinite continuous linear self-adjoint operator. Then, by positive semi-definiteness of T, it follows that f is convex function. For every $y \in F$ we have

$$\begin{aligned} 0 \leq & \langle T\bar{x} + c, y - \bar{x}\rangle = \\ & = \lim_{t\downarrow 0} \frac{f(\bar{x} + t(y - \bar{x})) - f(\bar{x})}{t} \leq f(y) - f(\bar{x}). \end{aligned}$$

It follows that $\bar{x}$ is a local solution of (QP). Hence $\bar{x}$ is a local solution of (QP) if and only if $\bar{x}$ is a solution of (sAVIω_0) and (3) is sufficient condition for $\bar{x}$ to be a local solution of (QP).

Theorem 4.2. *Consider the problem* (QP) *where* $\langle x, Tx\rangle$ *is a Legendre form. Let* $\bar{x} \in \mathcal{H}$ *and* $\lambda \in \mathbb{R}^m$ *be such that the system* (3) *is satisfied, and suppose that the condition* (14) *holds, then the point* $\bar{x}$ *is a locally unique solution of* (QP) *and for all* $\bar{x}(\omega) \in S(\omega)$ *in a neighborhood of* $\bar{x}$*, we have* $\|x(\omega) - \bar{x}\| = O(\|\omega - \omega_0\|)$.

Proof. To prove the theorem, by Theorem 3.3 and Theorem 3.1 in [3], it suffices to verify that $\bar{x}$ is a locally unique solution for (QP). Indeed, suppose that the point $\bar{x}$ is not a locally unique solution for (QP). Then there exists a sequence of feasible points x_k, converging to $\bar{x}$, $x_k \neq \bar{x}$, such that

$$f(x_k) \leqslant f(\bar{x}). \tag{17}$$

Set $t_k := \|x_k - \bar{x}\|$ and $h_k := \dfrac{x_k - \bar{x}}{t_k}$. We have $t_k > 0$, $\|h_k\| = 1$ and

$$\begin{aligned} &\langle T_i\bar{x} + c_i, h_k\rangle = \\ &= \frac{1}{t_k}\{g_i(x_k) - g_i(\bar{x}) - \frac{1}{2}\langle x_k - \bar{x}, T_i(x_k - \bar{x})\rangle\} \leqslant 0 \end{aligned}$$

for all $i \in I(\bar{x})$. Put $C(\bar{x}) = \{h \in \mathcal{H} \mid \langle T\bar{x} + c, h\rangle = 0, \langle T_i\bar{x} + c_i, h\rangle \leqslant 0,\ i \in I(\bar{x})\}$. It follows from Hoffman's lemma (see, for instance, [7, Theorem 3]) that

$$\begin{aligned} &dist(h_k, C(\bar{x})) \leqslant \beta\Big([\langle T\bar{x} + c, h_k\rangle]_+ + \\ &+ \sum_{i\in I(\bar{x})} [\langle T_i\bar{x} + c_i, h_k\rangle]_+\Big) = \beta\Big([\langle T\bar{x} + c, h_k\rangle]_+\Big) \end{aligned}$$

where, $\beta > 0$ depending on $T\bar{x} + c$ and $T_i\bar{x} + c_i$. By (17) and

$$f(x_k) - f(\bar{x}) = t_k\langle T\bar{x} + c, h_k\rangle + \frac{t_k^2}{2}\langle h_k, Th_k\rangle,$$

it follows that

$$t_k\langle T\bar{x} + c, h_k\rangle \leqslant -\frac{t_k^2}{2}\langle h_k, Th_k\rangle. \tag{18}$$

Since $|\langle h_k, Th_k\rangle| \leqslant \|T\|\|h_k\|^2 = \|T\|$, it follows that $-\dfrac{t_k}{2}\langle h_k, Th_k\rangle \to 0$ as $k \to \infty$. Combining this with (18) we have that $t_k\big(\langle T\bar{x}+c, h_k\rangle\big) \leq o(t_k)$, and hence there exists $\hat{h}_k \in C(\bar{x})$ such that $\hat{h}_k \to h_k$, and hence $\|\hat{h}_k\| = 1$.

Observe that

$$\begin{aligned} \langle \hat{h}_k, T\hat{h}_k\rangle - \langle h_k, Th_k\rangle &= \langle \hat{h}_k + h_k, T(\hat{h}_k - h_k)\rangle \\ &\leq \|\hat{h}_k + h_k\|\|T\|\|\hat{h}_k - h_k\|. \end{aligned}$$

From this and $\|\hat{h}_k - h_k\| \leqslant \beta([\langle T\bar{x} + c, h_k\rangle]_+)$ we deduce that

$$\langle \hat{h}_k, T\hat{h}_k\rangle - \langle h_k, Th_k\rangle \leqslant 2\beta\|T\|([\langle T\bar{x} + c, h_k\rangle]_+). \tag{19}$$

Consequently,

$$\begin{aligned} f(x_k) =& f(\bar{x}) + t_k\langle T\bar{x} + c, h_k\rangle + \frac{t_k^2}{2}\langle h_k, Th_k\rangle \\ \geqslant& f(\bar{x}) + t_k\langle T\bar{x} + c, h_k\rangle + \frac{t_k^2}{2}\langle \hat{h}_k, T\hat{h}_k\rangle \\ &- t_k^2\beta\|T\|([\langle T\bar{x} + c, h_k\rangle]_+). \end{aligned}$$

Since $\langle T\bar{x} + c, h\rangle \geqslant 0$ for all $h \in T_F(\bar{x})$, we have $\langle T\bar{x} + c, h_k\rangle \geqslant 0$ for k large enough. It follows that for k large enough

$$\begin{aligned} f(x_k) - f(\bar{x}) \geqslant& t_k\langle T\bar{x} + c, h_k\rangle + \frac{t_k^2}{2}\langle \hat{h}_k, T\hat{h}_k\rangle \\ &- t_k^2\beta\|T\|([\langle T\bar{x} + c, h_k\rangle]_+) > 0, \end{aligned}$$

which contradicts (17). The proof is complete. $\square$

Corollary 4.1. *Consider the problem* (QP) *Let* $\dim\mathcal{H} < \infty$ *and suppose that there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ *and* $\bar{x} \in \mathcal{H}$ *such that the* (3) *holds. Then, if condition* (14) *is satisfied, then the point* $\bar{x}$ *is a locally unique solution of* (QP) *and for all* $x(\omega) \in S(\omega)$ *in a neighborhood of* $\bar{x}$*, we have that*

$$\|x(\omega) - \bar{x}\| = O\|\omega - \omega_0\|.$$

Proof. Since the space $\mathcal{H}$ is finite dimensional, any quadratic form $Q : \mathcal{H} \to \mathbb{R}$ is a Legendre form. Hence the assertion follows from Theorem 4.2. $\square$

5 CONCLUSIONS

By using the basic analysis tools, such as tangent cone, critical cone, conditions for particular points to be solution and stability of the solutions for the class of semi-affine variational inequalities problem in the infinite-dimensional Hilbert space are obtained. We obtained stability of solutions for the parametric quadratically constrained quadratic programming problems in the infinite-dimensional Hilbert space.

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