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AUXILIARY FUNCTION METHOD IN SOLVING SOME INTEGRATION PROBLEMS USING AVERAGE VALUE THEOREM

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Abstract:

The selection of the auxiliary function is always a difficult problem if the learners do not thoroughly understand the theorems about the mean value, as well as the experience of solving this type of math. In addition, the author has many articles, personal remarks on the above problem. The author has analyzed and evaluated the solution method, through the collection, selection and classification of a number of typical problems, with the aim of perfecting the selection of sub-functions in the integral problem for students, helping students study better with certain themes, research and achieve good results in Student Math Olympiads, especially for Tan Trao's students. In addition, the author also has articles and personal evaluations during the teaching process. The result of the article is to analyze and clarify how to use the mean value theorem for problems related to integration. Collecting a suitable system of exercises, contributing to helping students master the form of math in the process of learning and researching. The completed article has practical significance for Tan Trao's students, especially students in the Math Olympiad team and students loving math. It is a topic that students can continue collecting and completing into meaningful learning materials.

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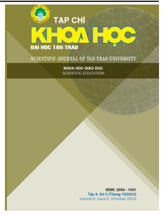

PHƯƠNG PHÁP HÀM PHỤ TRONG GIẢI MỘT SỐ BÀI TOÁN TÍCH PHÂN BẰNG ĐỊNH LÍ GIÁ TRỊ TRUNG BÌNH

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Thông tin bài viết

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Hàm số, vi phân, tích phân, giá trị trung bình, Olympic sinh viên.

Tóm tắt

Thông qua việc sưu tầm, lựa chọn, phân loại một số bài toán tiêu biểu, tác giả phân tích, đánh giá phương pháp giải, với mục đích hoàn thiện cách chọn hàm phụ trong bài toán tích phân cho sinh viên, giúp sinh viên có một chuyên đề học tập, nghiên cứu và đạt kết tốt trong các kỳ thi Olympic toán sinh viên, đặc biệt đối với sinh viên của Trường Đại học Tân Trào. Ngoài ra, tác giả cũng có những bài, những đánh giá của cá nhân trong quá trình giảng dạy. Kết quả của bài báo là đã phân tích, làm rõ cách sử dụng định lí giá trị trung bình cho các bài toán liên quan đến tích phân. Sưu tầm hệ thống bài tập phù hợp, góp phần giúp sinh viên nắm vững dạng toán trong quá trình học tập và nghiện cứu. Bài báo hoàn thiện có ý nghĩa thực tiễn đối với sinh viên Trường Đại học Tân Trào, đặc biệt đối với sinh viên trong đội tuyển Olympic Toán và những sinh viên yêu thích môn Toán. Là một chuyên đề mà sinh viên có thể tiếp tục sưu tầm, hoàn thiện thành tài liệu học tập có ý nghĩa.

1. Introduction

Differentiation and integration are one of the interesting and difficult topics in high school and university calculus programs, especially the application of the mean value theorem. These types of math are highly applied in the National High School Exams, excellent students and math Olympiads for students. Some of these types of math have been mentioned in some documents and textbooks but are not yet systematic, especially how to choose auxiliary functions. There has been a number of articles and scientific conferences that have also mentioned this topic. However, through monitoring the results of student math Olympiads, especially for students at Tan Trao University, they do not do well in the above types of math. In the article, the author focuses on exploiting a form of mathematics that applies the mean value theorem in differentiation and integration, orienting the method of finding appropriate auxiliary functions in problems related to integration, thereby Students recognize and solve these types of math well in exams, and proactively build a system of exercises during the learning and research process.

2. Research Method

Collecting, selecting, and classifying a number of typical problems, the author analyzes and evaluates solution methods, instructing learners to perfect how to choose auxiliary functions in integral problems using theorems on mean value. jar.

In addition, the author also has articles and personal assessments about his experiences in the teaching process.

Compared to some documents written on this topic, the author finds that the way of analyzing the problem leads to more clarity on the need to use relevant knowledge. In addition, whether the problem can be specialized or generalized, learners can rely on it to master the whole system of problems, and can adjust, add or remove assumptions to create new problems.

3. Result and discussion

3.1. Preparatory knowledge

First of all, we need to recall some theorems about differentiability and integrability related to the content of the topic. These are basic theorems, proven in the shown documents. Readers need to refer to how to prove these theorems, which is also a theoretical method in the process of solving problems.

3.1.1. Theorem of differentiability [6],[10]

Theorem 1. (Fermat's theorem). Given the function $f:[a;b]\to\mathbb{R}$ is continuous in $[a;b]$ and reaches maximum at point $x_0\in(a;b)$, differentiable at point x_0, then $f'(x_0)=0$.

Theorem 2. (Rolle's theorem). Given the function $f:[a;b]\to\mathbb{R}$ continuous in $[a;b]$ and differentiable in $(a;b)$ satisfy $f(a)=f(b)$. Then exists a point $c\in(a;b)$ such that $f'(c)=0$.

Theorem 3. (Lagrange's theorem). Given the function $f:[a;b]\to\mathbb{R}$ continuous in $[a;b]$ and differentiable in $(a;b)$. Then exists a point $c\in(a;b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Theorem 4 (Cauchy's theorem). Given the functions $f,g:[a;b]\to\mathbb{R}$ are continuous in $[a;b]$ and differentiable in $(a;b)$, $g'(x)\neq 0,\forall x\in(a;b)$.Then exists a point $c\in(a;b)$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

3.1.2. Theorem of integrability [6],[10]

Theorem 5. If $f(x)$ is a intergrability function in $[a;b]$ and continuous for all points $x\in[a;b]$ then the function $F(x)=\int_a^x f(t)dt$ differentiable at x and $F'(x)=\left(\int_a^x f(t)dt\right)'=f(x)$.

General we have

$$F'(x)=\left(\int_{u(x)}^{v(x)} f(t)dt\right)'=v'(x)f(v(x))-u'(x)f(v(x))\cdot$$

Theorem 6 (Mean value theorem). If the f is a intergrability function in $[a;b]$ and $m\le f(x)\le M$ for all $x\in[a;b]$, then exist a number $\alpha\in[m;M]$ such that $\int_a^b f(x)dx=\alpha(b-a)$.

Consequence. If the function f continuous in $[a;b]$ then exist at least a point $c\in[a;b]$ such that $\int_a^b f(x)dx=f(c)(b-a)$.

Theorem 7. (Mean value theorem open). Suppose f and g are two intergrability functions in $[a;b]$. If $m\le f(x)\le M$ for all $x\in[a;b]$ and $g(x)$ do not change the sign in $[a;b]$, then exist at least a point $\alpha\in[m;M]$ such that $\int_a^b f(x)g(x)dx=\alpha\int_a^b g(x)dx$.

Consequence. If the function f is continuous in $[a;b]$ and g is a intergrability functions in $[a;b]$ then exist at least a point $c\in[a;b]$ such that $\int_a^b f(x)g(x)dx=f(c)\int_a^b g(x)dx$.

3.2. Illustrative examples

Example 1. Given the continuous function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=\int_0^1 xf(x)dx$. Prove that always exists point $c \in (0;1)$ such that $\int_0^c f(x)dx = 0$ [9].

Analyze to find solutions. *From the problem, we see knowledge related to theorem 2. Therefore we need to choose an auxiliary function* $g(t)$ *satisfies theorem 2 and in the conclusion of the theorem* $g'(c) = 0$ *leads to the desired equality* $\int_0^c f(x)dx = 0$. *This is suggested by the hypothesis* $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$.

Solution. *Consider the function* $g(t) = t\int_0^t f(x)dx - \int_0^t xf(x)dx,\ t \in [0;1]$. *We have*

$$g(0) = g(1) = 0 \text{ and}$$

$$g'(t) = \left[t\int_0^t f(x)dx - \int_0^t xf(x)dx\right]' = \int_0^t f(x)dx + tf(t) - tf(t) = \int_0^t f(x)dx.$$

Hence the function $g(t)$ *satisfies theorem 1.2, so the exists number* $c \in (0;1)$ *such that* $g'(c) = 0 \Leftrightarrow \int_0^c f(x)dx = 0$.

Comment. *You can use Lagrange's theorem or Cauchy's theorem. But note that for each theorem, the choice of function to satisfy the theorem's assumptions is different and not always immediate. We need to try and adjust to hit the mark. For the example above, we choose the function* $g(t)$ *satisfy Rolle's theorem* $g(0) = g(1)$ *to have* $f'(c) = \int_0^c f(x)dx = 0$.

Example 2. *Given the continuous fuction* $f:[0;1] \to \mathbb{R}$ *such that* $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. *Prove that for every real number* a *,then there always exists the number* $c \in (0;1)$ *so that* $f(c) = a\int_0^c f(x)dx$ [9].

Analyse to find solution. *We see that the assumption of the problem does not change compared to example 1, but the equality needs to be proven differently, so it is necessary to adjust the auxiliary function so that there is equality* $f(c) = a\int_0^c f(x)dx \Leftrightarrow k\left(f(c) - \int_0^c f(x)dx\right) = 0$.

To exist the original function after taking the derivative, people often choose a function related to the exponential function, from which we can find the auxiliary function.

Solution. With the given assumption, according to the results of example 1, there exists $\alpha \in (0;1)$ *so that* $\int_0^\alpha f(x)dx = 0$. *For* $t \in [0;1]$, *consider the fuction* $g(t) = e^{-at}\int_0^t f(x)dx$. *We have* $g(0) = g(\alpha) = 0$ *and* $g'(t) = e^{-at}\left[f(t) - a\int_0^t f(x)dx\right]$. *According to theorem 1.2,then there exists a number* $c \in (0;\alpha) \subset (0;1)$ *so that* $g'(c) = 0$. *It means* $g'(c) = e^{-ac}\left[f(c) - a\int_0^c f(x)dx\right] = 0$.

Hence, $f(c) = a\int_0^c f(x)dx$.

Comment. *If* $a = 0$ *the problem is always correct. If* $a = 2018$ *we have 2018 Student Math Olympiad exam [5]. From the above example, we notice that selection function* $g(t)$ *by the exponential function and have seen its effects:* $\left[a^{kx}.f(x)\right]' = a^{kx}\left[kf(x) + f'(x)\right]$. *Similar exercises will be found in the following sections and self-practice exercises, gradually generalized from the above examples. We can develop the above problem in another direction as follows: Given the continuous function* $f:[0;1] \to \mathbb{R}$ *satisfy* $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. *Prove that for every real* *number M, there exist the number $c\in(0;1)$ so that $M=\dfrac{f(c)}{\int_0^c f(x)dx}$ [7].*

Example 3. *Given the linear function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=\int_0^1 xf(x)dx=1$. Prove that exist the point $c\in(0;1)$ so that $f'(c)=6$.*

Analyse to find solution. *Because f is the linear function therefore from the assumption $\int_0^1 f(x)dx=\int_0^1 xf(x)dx=1$, we will find the function f. After the problem is easily solved.*

Solution. The first we find the function f: Because f is the linear therefore $f(x)=ax+b,\ a\neq 0$. It takes by assumption

$$\begin{cases}\int_0^1 f(x)dx=1\\ \int_0^1 xf(x)dx=1\end{cases}\Leftrightarrow\begin{cases}\int_0^1 (ax+b)dx=1\\ \int_0^1 x(ax+b)dx=1\end{cases}$$

$$\Leftrightarrow\begin{cases}\left(\frac{1}{2}ax^2+bx\right)\Big|_0^1=1\\ \left(\frac{1}{3}ax^3+\frac{1}{2}bx^2\right)\Big|_0^1=1\end{cases}\Leftrightarrow\begin{cases}a=6\\ b=-2\end{cases}.$$

Hence, $f(x)=6x-2$ and $f'(x)=6,\ \forall x\in\mathbb{R}$, so of course there exists a point $c\in(0;1)$ so that $f'(c)=6$.

Comment. *In the general case, we don't know f is the linear function, by the exemple 2 we still choose (simplest function) satisfy the assumption, that is the linear fuction and we find $f(x)=6x-2$. Specifically: Given the differentiable function $f:[0;1]\to\mathbb{R}$ so that $\int_0^1 f(x)dx=\int_0^1 xf(x)dx=1$. Prove that there exist the point $c\in(0;1)$ so that $f'(c)=6$ [3].*

Solution. Consider the function $g(x)=6x-2$ we have $\int_0^1 f(x)dx=\int_0^1 xf(x)dx=1$ (according to the above result), therefore $\int_0^1[f(x)-g(x)]dx=0$. Consider the continuous function $h(x)=f(x)-g(x)$ in $[0;1]$. We have $\int_0^1 h(x)dx=0$, so that doesn't happen $h(x)<0,\forall x\in(0;1)$ or $h(x)>0,\forall x\in(0;1)$, it means the equation $h(x)=0$ must has at least one solution in $(0;1)$

Suppose $h(x)=0$ has a solution $x=a\in(0;1)$, then there are two possibilities:

- If $h(x)<0,\forall x\in(0;a)$ then $h(x)>0,\forall x\in(a;1)$, so

$$\int_0^1 xf(x)dx-1=\int_0^1 xf(x)dx-\int_0^1 xg(x)dx=\int_0^1 x[f(x)-g(x)]dx=\int_0^1 xh(x)dx$$

$$=\int_0^a xh(x)dx+\int_a^1 xh(x)dx>\int_0^a ah(x)dx+\int_a^1 ah(x)dx=a\left[\int_0^a h(x)dx+\int_a^1 h(x)dx\right]$$

$=a\int_0^1 h(x)dx=0\Rightarrow\int_0^1 xf(x)dx>1$, contradict the hypothesis.

- If $h(x)>0,\forall x\in(0;a)$ then $h(x)<0,\forall x\in(a;1)$, so:

$$\int_0^1 xf(x)dx-1=\int_0^1 xf(x)dx-\int_0^1 xg(x)dx=\int_0^1 x[f(x)-g(x)]dx=\int_0^1 xh(x)dx$$

$$= \int_0^a xh(x)dx + \int_a^1 xh(x)dx < \int_0^a ah(x)dx + \int_a^1 ah(x)dx = a\left[\int_0^a h(x)dx + \int_a^1 h(x)dx\right]$$

$= a\int_0^1 h(x)dx = 0 \Rightarrow \int_0^1 xf(x)dx < 1$, contradict the hypothesis.

Hence, $h(x)=0$ has at least two solutions in the $(0;1)$. Suppose two that solutions are $a,b \in (0;1)$ and $a<b$. We have $h(a)=h(b)=0$ should $f(b)-f(a)=g(b)-g(a)$.

By the Lagrange's theorem, there exist the number $c \in (a;b) \subset (0;1)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{g(b)-g(a)}{b-a} = 6.$$

Command. *From the above problem we have , if given the asumption* $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$ *and add more one condition then can be found the function* $f(x)$. *Readers can refer to the author's article published in 2022 [8]: Find the solution of the differential-integral functional equation using the identity method: Find the continuous function* $f(x)$ in $[0;1]$, *satisfy*

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1 \text{ and } \int_0^1 f^2(x)dx = 4.$$

Consider the following identity:

$$\left[f(x)+ax+b\right]^2 = 0 \Leftrightarrow f'^2(x) + 2axf(x) + 2bf(x) + a^2x^2 + 2abx + b^2x^2 = 0.$$

Therefore $\left[f(x)+ax+b\right]^2 = 0 \Leftrightarrow f'^2(x) + 2axf(x) + 2bf(x) + a^2x^2 + 2abx + b^2x^2 = 0$

$$\Leftrightarrow \int_0^1 f^2(x)dx + 2a\int_0^1 xf(x)dx + 2b\int_0^1 f(x)dx + \int_0^1 \left(a^2x^2 + 2bx + b^2\right)dx = 0$$

$\Leftrightarrow 4 + 2(a+b) + \frac{a^2}{3} + ab + b^2 = 0$. We need finding a,b such that:

$$\int_0^1 \left[f(x)+ax+b\right]^2 dx = 0, \text{ or } 4 + 2(a+b) + \frac{a^2}{3} + ab + b^2 = 0$$

$\Leftrightarrow a^2 + (3b+6)a + 3b^2 + 6b + 12 = 0$.

Solving the equation for unknown a, we find $b=2,\ a=-6$.

Therefore from $\int_0^1 \left[f(x)-6x+2\right]^2 dx = 0 \Rightarrow f(x) = 6x-2, \forall x \in [0;1]$.

Exxample 4. Given the function f is differentiable twice in ℝ satisfy $f(0)=f(1)=253$ and $\min\limits_{x\in[0;1]} f(x) = \frac{1}{8}$. Prove that $\max\limits_{x\in[0;1]} f''(x) \ge 2023$ [9].

Analyse to find solution. *Problem related to the minimum and maximum value of the function, that suggests we use the Fermat's theorem. The problems related to second derivative, therefore we can calculate directly or use an auxiliary function, or Taylor expansion to second order.*

Solution. Let $c \in [0;1]$ such that $f(c) = \min\limits_{x\in[0;1]} f(x) = \frac{1}{8}$. We have $c \neq 0$ và $c \neq 1$, because if so $f(0)=f(1)=253>\frac{1}{8}$, unsatisfying. According to Fermat's theorem we have $f'(c)=0$.

Taylor expansion the function $f(x)$ in the neighbourhood of the point c we have

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''[c+\varepsilon(x-c)]}{2}(-c)^2,\ 0<\varepsilon<1.$$

At the $x=0$ we have $f(0)=f(c)+f'(c)(-c)+\frac{f''[c+\varepsilon_1(-c)]}{2}(-c)^2,\ 0<\varepsilon_1<1$.

Dedue $f''[c+\varepsilon_1(-c)]=\frac{2023}{4c^2}$.

Similar, at the $x=1$ we have $f''[c+\varepsilon_2(1-c)]=\frac{2023}{4(1-c)^2}$.

Beside, according to unequality AM-GM we have $c(1-c)\le\left(\frac{c+1-c}{2}\right)^2=\frac{1}{4}$.

Hence $f''[c+\varepsilon_1(-c)].f''[c+\varepsilon_2(1-c)]=\frac{2023^2}{16c^2(1-c)^2}\ge 2023^2$.

So $2023^2\le f''(c)[c+\varepsilon_1(-c)].f''[c+\varepsilon_2(1-c)]\le\left[\max_{x\in[0;1]} f''(x)\right]^2$. Deduce $\max_{x\in[0;1]} f''(x)\ge 2023$.

Command. *The above problem is a special case of the following general problem: Given the function f be differentiable twice in satisfy $f(0)=f(1)=M$ and $\min_{x\in[0;1]} f(x)=m<M$. Prove that $\max_{x\in[0;1]} f''(x)\ge 8(M-m)$. The solution is similar to the one above [7].*

Example 5. *Given the function f has the first derivative in $[1;2]$ and has the second derivative in $(1;2)$, satisfy $\int_1^2 f(x)dx=2f(2)-f(1)-1, f'(1)=1$.*

Prove that equaltion $xf''(x)+f'(x)=0$ is always has the solution in the $(1;2)$. [7].

Analyse to find the solution. *We have in the problem, the integral value is related to the limit of integration, therefore, we think of using the mean-integral theorem 2.2 and the auxiliary function will be related to the first derivative.*

Solution. We notice that the function $g(x)=xf'(x)$ has derivative in the $[1;2]$, so it is also continuous in it. According to the mean value theorem 2.2, exitst the number $c\in(1;2)$ such that:

$$\int_1^2 g(x)dx=g(c)(2-1)=cf'(c).$$

In the integral $\int_1^2 xf'(x)dx$, set $\begin{cases}x=u\\ f'(x)dx=dv\end{cases}\Rightarrow\begin{cases}du=dx\\ v=f(x)\end{cases}$, we have:

$$\int_1^2 xf'(x)dx=xf(x)\Big|_1^2-\int_1^2 f(x)dx$$

$$=2f(2)-f(1)-\int_1^2 f(x)dx=1.$$ Hence

$$cf'(c)=1.$$

Consider the function $h(x)=xf'(x)-1$ then the function h is continuous in $[1;2]$ and it is differentiable in $(1;2)$. We have: $h(1)=h(c)=0,\ h'(x)=xf''(x)+f'(x)$.

Accorđing to Rolle theorem, exist the real number $\alpha\in(1;c)\subset(1;2)$ such that $h'(\alpha)=0$.

I.e. the equaltion $xf''(x)+f'(x)=0$ always has the solution in $(1;2)$.

Command. *The above problem is a special of the general problem below: Given the function* f *has the first derivative in* $[a;b]$ *and the second derivative in* $(a;b)$, *satisfying* $\int_a^b f(x)dx=bf(b)-af(a)-1,\ af'(a)=1$. *Prove that the equaltion* $xf''(x)+f'(x)=0$ *has always the solution in* $(a;b)$. *The solution of the general problem is similar the above problem [7].*

Example 6. *Given the* $f:[0;1]\to\mathbb{R}$ is the countinuous differentiable function in $[0;1]$ and satisfying $\int_0^1\left[f'(x)+(1-2x)f(x)\right]dx=\frac{10115}{6}$.

Prove that existing the $c\in(0;1)$ such that $f'(c)=2023$.

Analyse to find the solution. *According to the problem we have, if we calculate the integral* $\int_0^1(1-2x)f(x)dx$ *by the method of integration by parts then will create a related expression, after that we apply the mean value theorem 2.3 to achieve the purpose.*

Solution. Consider the intergral $I=\int_0^1(1-2x)f(x)dx$. Using method of integration by parts:

Set $\begin{cases}u=f(x)\\(1-2x)dx=dv\end{cases}\Rightarrow\begin{cases}du=f'(x)dx\\v=x-x^2\end{cases}$,

we have:

$$I=\left(x-x^2\right)f(x)\Big|_0^1-\int_0^1\left(x-x^2\right)f'(x)dx=\int_0^1\left(x-x^2\right)f'(x)dx.$$

Hence $\int_0^1\left[f'(x)+(1-2x)f(x)\right]dx=\frac{10115}{6}\Leftrightarrow\int_0^1\left(x^2-x+1\right)f'(x)dx=\frac{10115}{6}$.

Applying the mean theorem 2.3 for the interal with the function $f'(x)$ and the function $g(x)=x^2-x+1>0$, then for all $x,\ \exists c\in(0;1)$ such that :

$$\frac{10115}{6}=f'(c)\int_0^1\left(x^2-x+1\right)dx=\frac{5}{6}f'(c),\text{ or }f'(c)=2023.$$

Command. *The above problem is a special case of the general problem below: Given* $f:[0;1]\to\mathbb{R}$ *is the continuous differentiable in the* $[0;1]$ *and satisfying:*

$$\int_0^1\left[f'(x)+(1-2x)f(x)\right]dx=m.$$

Prove that exist $c\in(0;1)$ *such that* $f'(c)=\frac{6m}{5}$ *[7].*

Example 7. Prove that exist the real number $x\in(0;1)$ such that:

$$\int_x^1 \frac{t^{2000}}{(1+t)(1+t^2)...(1+t^{2001})}dt = \frac{x^{2001}}{(1+x)(1+x^2)...(1+x^{2001})}$$ [1].

Analyse to find the solution. *From the problem we have:*

$$\int_x^1 \frac{t^{2000}}{(1+t)(1+t^2)...(1+t^{2001})}dt - \frac{x^{2001}}{(1+x)(1+x^2)...(1+x^{2001})} = 0$$

Therefore we only consider two functions with appropriate exponents, after that use the mean value theorem.

Solution. Consider the functions:

$$f(t) = \frac{t^{2000}}{(1+t)(1+t^2)...(1+t^{2001})}, t \in [0;1] \text{ and } F(x) = x\int_x^1 f(t)dt, x \in [0;1].$$

Therefore $F(0) = F(1) = 0$ and $f(t)$ continuous in $[0;1]$ so that $F(x)$ is differentiable in $(0;1)$. According to theorem 1.2, existing $x \in (0;1)$ to $F'(x) = 0$.

We have: $F'(x) = \int_x^1 f(t)dt - xf(x)$.

$$F'(x) = 0 \Leftrightarrow \int_x^1 \frac{t^{2000}}{(1+t)(1+t^2)...(1+t^{2001})}dt - \frac{x^{2001}}{(1+x)(1+x^2)...(1+x^{2001})} = 0.$$

Hence, $\int_x^1 \frac{t^{2000}}{(1+t)(1+t^2)...(1+t^{2001})}dt = \frac{x^{2001}}{(1+x)(1+x^2)...(1+x^{2001})}$.

Command. *From the above examples, readers may be already get oriented the way use the mean value and choose the auxiliary function. Similar problems and gradually expanded from specific problems will be given in the self-practice exercises section.*

3. Self-practice exercises

Exercise 1. Given the continuous function $f:[0;1] \to \mathbb{R}$ and $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. Prove that exist the point $\alpha \in (0;1)$ such that $\int_0^\alpha xf(x)dx = 0$ [9].

Directive. Consider the function $h(t) = t\int_0^t f(x)dx - \int_0^t xf(x)dx,\ t \in [0;1]$. Using the theorem 1.2.

Exercise 2. Given the continuous function $f:[0;1] \to \mathbb{R}$ and $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. Prove that for every number $a \in \mathbb{R}$, existing the point $c \in (0;1)$ such that $cf(c) = a\int_0^c xf(x)dx$ [9].

Directive. For $t \in [0;1]$, consider the function $g(t) = e^{-at}\int_0^t xf(x)dx$. According to above problem, exist the point $\alpha \in (0;1)$ such that $\int_0^\alpha xf(x)dx = 0$. Using the theorem 1.2.

Exercise. Given the continuous function $f:[0;1] \to \mathbb{R}$ and $\int_0^1 f(x)dx = \int_0^1 xf(x)dx$. Prove that for every the number $a \in \mathbb{R}$, exist the point $c \in (0;1)$ such that $f(c) = af'(c)\int_0^c f(x)dx$ [5].

Directive. For the number $t\in[0;1]$, consider the function $g(t)=e^{-af(t)}\int_0^t f(x)dx$, exist the number $\alpha\in(0;1)$ such that $\int_0^\alpha f(x)dx=0$. Using the theorem 1.2.

Command. If we take $a=2018$, that is invited to the 2018 Student Math Olympiad [5].

Exercise 4. Given the differentiable function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=\int_0^1 xf(x)dx$. Prove that for every the number $a\in\mathbb{R}$, existing the point $c\in(0;1)$ such that: $cf(c)=af'(c)\int_0^c xf(x)dx$ [9].

Directive. For the number $t\in[0;1]$, consider the function $g(t)=e^{-af(t)}\int_0^t f(x)dx$. According to above problem, existing the number $\alpha\in(0;1)$ such that $\int_0^\alpha xf(x)dx=0$. Using the theorem 1.2.

Exercise 5. Given the differentiable function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=\int_0^1 xf(x)dx$. Prove that for every the number $a\in\mathbb{R}$, existing the point $c\in(0;1)$ such that: $cf(c)+a\int_0^c f(x)dx=0$ [9].

Directive. Consider the function $F(x)=\int_0^x f(t)dt$ and $g(x)=x^aF(x)$. Using the theorem 1.2.

Exercise 6. Given the continuous function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=0$. Prove that for every the real number a, existing the number $c\in(0;1)$ such that: $f(c)=a\int_0^c f(x)dx$ [5].

Exercise 7. Given the continuous function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=0$. Prove that for every the real number a, existing the number $c\in(0;1)$ such that: $(1-c)f(c)=c\int_0^c f(x)dx$ [9].

Exercise 8. Given the differentiable function f in $[0;1]$ and $\int_0^1 f(x)dx=0$. Prove that for every the real number a, existing the number $c\in(0;1)$ such that $f(c)=af'(c)\int_0^c f(x)dx$ [9].

Exercise 9. Given the continuous function $f:[0;1]\to\mathbb{R}$ and $\int_0^1 f(x)dx=0$. Prove that existing the number $c\in(0;1)$ such that $\int_0^c xf(x)dx=0$ [7].

Exercise 10. Given the number $a\in(0;1)$. Suppose the continuous function f in the $[0;1]$ and $f(0)=f(1)=0$. Prove that exist the number $b\in[0;1]$ such that $f(b)=f(b-a)$ or $f(b)=f(b+a-1)$ [9].

Exercise 11. Given the continuous f in $\left[0;\frac{\pi}{2}\right]$ and $f(0)>0,\ \int_0^{\frac{\pi}{2}} f(x)dx<1$. Prove that equaltion $f(x)=\sin x$ has at least one solution in $\left(0;\frac{\pi}{2}\right)$ [9].

Exercise 12. Given the second differentiable function f in $[0;1]$ and $f''(x)\ge 0,\forall x\in[0;1]$. Prove that $2\int_0^1(1-x)f(x)dx\le\int_0^1 f(x^2)dx$ [7].

Exercise 13. Given the differentiable twice function f in $[0;1]$ and $f''(x)\ge 0,\forall x\in[0;1]$. Prove that: $2\int_0^1 f(x)dx\ge 3\int_0^1 f(x^2)dx-f(0)$ [2].

Exercise 14. Given the continuous function f in $[0;2012]$ and satisfying:

$$f\left(x\right)+f\left(2012-x\right)=0,\ \forall x\in\left[0;2012\right].$$

Prove that $\int_0^{2012} f\left(x\right)dx=0$ and the equaltion $\left(x-2012\right)f\left(x\right)=2012\int_0^{2012-x} f\left(t\right)dt$ has the solution in $\left(0;2012\right)$ [4].

4. Conclusion

Through monitoring Olympic exams for pupils and students, the author has found that the types of math using the mean value from basic to complex, possibly integrating from basic math problems, it is possible to synthesize each specific form to form a complete topic for learners to improve their knowledge, to be able to generalize and create a system of exercises in learning and research.

During the process of collecting and synthesizing documents, due to limited capacity and time, some of the results of the new topic stop at initial conclusions, some issues of the topic may not have been developed yet. The depth and method may not be optimal. Therefore, we hope to receive the attention and comments of teachers and colleagues to improve the quality of the subject.

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