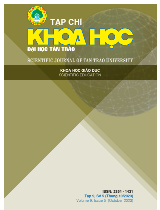
Vol 9. No 5_October 2023

TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

http://tckh.daihoctantrao.edu.vn/



CONTINUOUS REGULARIZATION METHOD FOR A COMMON MINIMUM POINT OF A FINITE SYSTEM OF CONVEX FUNCTIONALS

Tran Thi Huong

Thai Nguyen Univesity of Technology

Email address: tranthihuong@tnut.edu.vn

https://doi.org/10.51453/2354-1431/2023/910

Article info

Received: 12/8/2023

Revised:28/9/2023

Accepted: 16/10/2023

Keyword:

Monotone operators, hemi-continuous, Hilbert spaces, Gâteaux derivative, Tikhonov regularization.

Abstract:

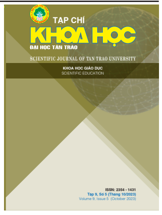
The concept of the ill-posed problem was introduced by Hadamard, a French mathematician in 1932 when he studied the effect of the boundary value problem on differential equations. Due to the unstability of the ill-posed problems, the numerical computation is difficult to do. Therefore, one of the main study directions for ill-posed problems is constructing stable methods to solve ill-posed problems such that when the error of the input data is smaller, the approximate solution is closer to the correct solution of the original problem. Although there are some known important results obtained in studying the regularization method for solving ill-posed problems, the improvement of the methods to increase their effectiveness always attracts the attention of many researchers. In this paper, we present a regularization method for a common minimum point of a finite system of Gâteaux differentiable weakly lower semi-continuous and properly convex functionals on real Hilbert spaces. And then, we give an application to illustrate the propose method.

Vol 9. No 5_October 2023

TẠP CHÍ KHOA HỌC ĐẠI HỌC TÂN TRÀO

ISSN: 2354 - 1431

http://tckh.daihoctantrao.edu.vn/



PHƯƠNG PHÁP HIỆU CHỈNH LIÊN TỤC CHO ĐIỂM CỰC TIỂU CHUNG CỦA HỌ HỮU HẠN CÁC HÀM LỒI

Trần Thị Hương

Đại học Kỹ thuật Công nghiệp - Đại học Thái Nguyên, Việt Nam

Địa chỉ email: tranthihuong@tnut.edu.vn

https://doi.org/10.51453/2354-1431/2023/910

Thông tin bài báo

Ngày nhận bài: 12/8/2023

Ngày sửa bài: 28/9/2023

Ngày duyệt đăng: 16/10/2023

Từ khóa:

Toán tử đơn điệu, hemi-liên tục, không gian Hilbert, đạo hàm Gâteaux, hiệu chỉnh Tikhonov.

Abstract:

Khái niệm bài toán đặt không chỉnh được nhà toán học người Pháp J. Hadamard đưa ra vào năm 1932, khi nghiên cứu ảnh hưởng của bài toán giá trị biên với phương trình vi phân. Do tính không ổn định của bài toán đặt không chỉnh nên việc giải số gặp nhiều khó khăn. Vì vậy, một trong những hướng nghiên cứu rất quan trọng về bài toán đặt không chỉnh đó là, xây dựng các phương pháp giải ổn định lớp bài toán này sao cho, khi sai số của dữ liệu đầu vào càng nhỏ thì nghiệm xấp xỉ càng gần với nghiệm chính xác của bài toán ban đầu. Tuy đã có nhiều kết quả đạt được cho việc nghiên cứu các phương pháp hiệu chỉnh giải bài toán đặt không chỉnh song việc cải tiến các phương pháp làm gia tăng tính hiệu quả của phương pháp là vấn đề thời sự và cấp thiết. Trong bài báo này, chúng tôi giới thiệu phương pháp hiệu chỉnh liên tục cho điểm cực tiểu chung của họ hữu hạn các hàm lồi, khả vi, nửa liên tục dưới yếu trong không gian Hilbert thực. Cuối cùng là ví dụ minh họa cho phương pháp đã đề xuất.

1. Introduction

Let H be a real Hilbert space with the scalar product and norm denoted by the symbols $\langle .;. \rangle$ and $\|.\|$, respectively, and let $\varphi_j(x), 0 \le j \le N,$ be weakly lower semi-continuous and property convex functionals on H.

Consider the problem: find an element $x_0 \in H$ such that

$$\varphi_j(x_0) = \inf_{x \in H} \varphi_j(x), \ \forall j = 0,1,...,N. \quad (1.1)$$

Set

$$S_j = \left\{ \tilde{x} \in H: \varphi_j(\tilde{x}) = \inf_{x \in H} \varphi_j(x) \right\}, S = \bigcap_{j=0}^{N} S_j.$$

Here, we suppose S ≠ ∅.

As we know that in [13], S coincides with the set of solutions of the following operator equation

$$A_j(x) = \theta, \qquad (1.2)$$

where θ is the zero element in H, and A_j is the Gâteaux derivative of the functional φ_j. Besides, S_j is a closed convex subset in H.

Without the additional conditions on A_j such as the strongly or uniformly monotone property each j-operator equation in (1.2) is ill-posed. Therefore, to find a solution of each j-operator equation in (1.2) we have to use stable methods. One of those methods is the Tikhonov regularization method [1] and is difined by

$$A_j^h(x) + \alpha(x - x_*) = \theta, \qquad (1.3)$$

where x_* is some element $H \setminus S_j$, $\alpha > 0$ is the parameter of regularization, A_j^h are the hemi-continuous monotone approximations for A_j in the sense

$$\left\| A_j(x) - A_j^h(x) \right\| \le hg(\|x\|), \ \forall x \in H \quad (1.4)$$

with the nonegative bounded function $g(t)$, $t \ge 0, h \to 0.$

Our problem: find

$$u(t): [t_0, +\infty) \to H, t_0 \ge 0,$$

such that $\lim_{t \to +\infty} u(t) = x, x \in S.$

To do this, consider the differential equation

$$\frac{du(t)}{dt} + \gamma(t)\left[\sum_{j=0}^{N} \alpha^j(t) A_j^{h(t)}(u(t)) + \alpha^{N+1}(t)(u(t) - x_*) \right] = \theta,$$

$$u(t_0) = u_0, \qquad (1.5)$$

with $x_* \notin S_0$, where u_0 is an element of H, $h(t)$, $\alpha(t) > 0$, $t \ge t_0 \ge 0$, $\alpha(t)$ is a convex decreasing function, $\gamma(t)$ is a nondecreasing positive and differentiable function, and

$$\lim_{t \to +\infty} \alpha(t) = \lim_{t \to +\infty} h(t) = 0,$$

$$\lim_{t \to +\infty} \frac{h(t)}{\alpha^{N+1}(t)} = \lim_{t \to +\infty} \frac{\alpha'(t)}{\alpha^{N+2}(t)\,\gamma(t)} = \lim_{t \to +\infty} \frac{\gamma'(t)}{\alpha^{N+1}(t)\,\gamma^2(t)} = 0. \qquad (1.6)$$

Note that equation (1.5) when $N = 0$ has the simple form

$$\frac{du(t)}{dt} + \gamma(t)\left[A_0^{h(t)}(u(t)) + \alpha(t)(u(t) - x_*) \right] = \theta,$$

$$u(t_0) = u_0. \qquad (1.7)$$

This equation is used in [2] with the case $\gamma(t) \equiv 1$, and $A_0^{h(t)} \equiv A$ in regularization ill-posed equations involving the accretive operator A.

2. Main results

First, consider the operator equation

$$\sum_{j=0}^{N} \alpha^j(\tau) A_j(x) + \alpha^{N+1}(\tau)(x - x_*) = \theta. \quad (2.1)$$

Since A_j are the maximal monotone operators defined on H [9], then the operator $\sum_{j=0}^{N} \alpha^j(\tau) A_j + \alpha^{N+1}(\tau) I$, where I is the identity operator in H, is maximal monotone [3,4,5] and coercive. Hence, equation (2.1) has a unique solution, denoted by $x_\alpha(\tau)$.

We have a result.

Theorem 2.1. $\lim_{\tau \to +\infty} x_\alpha(\tau) = x \in S$, *where*

$$\|x - x_*\| = \min_{x \in S} \|x - x_*\|.$$

Proof. From (2.1) it follows

$$\sum_{j=0}^{N} \alpha^j(\tau) \langle A_j(x_\alpha(\tau)), x_\alpha(\tau) - x \rangle + \alpha^{N+1}(\tau) \langle x_\alpha(\tau) - x_*, x_\alpha(\tau) - x \rangle = 0 \ \ \forall x \in S.$$

On the base of (1.2) and the monotone property of A_j we obtain

$$\langle x_\alpha(\tau) - x_*, x_\alpha(\tau) - x_* \rangle \le \langle x_\alpha(\tau) - x_*, x - x_* \rangle.$$

Thus,

$$\|x_\alpha(\tau)-x_*\| \le \|x-x_*\| \ \ \forall x\in S. \qquad (2.2)$$

Hence, $\{x_\alpha(\tau)\}$ is bounded. Let $x_\beta(\tau)$ weak convergence to $\bar{x}\in H,$ as $\tau\to+\infty.$ First, we prove that $\bar{x}\in S_0.$ Indeed, by virtue of the monotone property of A_0 and (2.1) we can write

$$\langle A_0(x), x-x_\beta(\tau)\rangle \ge \langle A_0(x_\beta(\tau)), x-x_\beta(\tau)\rangle$$

$$\ge \sum_{j=1}^{N}\beta^j(\tau)\langle A_j(x_\beta(\tau)), x_\beta(\tau)-x\rangle$$

$$+\beta^{N+1}(\tau)\langle x_\beta(\tau)-x_*, x_\beta(\tau)-x\rangle$$

$$\ge \sum_{j=1}^{N}\beta^j(\tau)\langle A_j(x), x_\beta(\tau)-x\rangle$$

$$+\beta^{N+1}(\tau)\langle x-x_*, x_\beta(\tau)-x\rangle \ \ x\in H.$$

By tending $\tau\to+\infty$ in the last inequality we have

$$\langle A_0(x), x-\bar{x}\rangle \ge 0 \ \ \forall x\in H.$$

Consequently, $\bar{x}\in S_0$ [13]. Now, we shall prove that $\bar{x}\in S_j,\ j=1,2,...,N.$ Indeed, from (1.2), (2.1) and the monotone property of A_0 it implies that

$$\langle A_1(x_\beta(\tau)), x_\beta(\tau)-x\rangle + \sum_{j=2}^{N}\beta^{j-1}(\tau)\langle A_j(x_\beta(\tau)), x_\beta(\tau)-x\rangle$$

$$+\beta^N(\tau)\langle x_\beta(\tau)-x_*, x_\beta(\tau)-x\rangle \le 0 \ \ \forall x\in S_0.$$

or

$$\langle A_1(x), x_\beta(\tau)-x\rangle + \sum_{j=2}^{N}\beta^{j-1}(\tau)\langle A_j(x), x_\beta(\tau)-x\rangle$$

$$+\beta^N(\tau)\langle x-x_*, x_\beta(\tau)-x\rangle \le 0.$$

After passing $\tau\to+\infty$, it gives

$$\langle A_1(x), \bar{x}-x\rangle \le 0 \ \ \forall x\in S_0.$$

Thus, $\bar{x}$ is a local minimizer for φ_1 on S_0. Since $S_0\cap S_1\ne\varnothing$, then $\bar{x}$ is also a global minimizer for φ_1, i.e., $\bar{x}\in S_1.$

Set $\tilde{S}_i=\bigcap_{k=0}^{i}S_k.$ Then, $\tilde{S}_i$ is also closed convex and $\tilde{S}_i\ne\varnothing.$

Now, suppose that we have proved $\bar{x}\in\tilde{S}_i$, and need to show that $\bar{x}$ belongs to S_{i+1}. Again, by virtue of (2.1) for $x\in\tilde{S}_i$, we can write

$$\langle A_{i+1}(x_\beta(\tau)), x_\beta(\tau)-x\rangle + \sum_{j=i+2}^{N}\beta^{j-(i+1)}(\tau)\langle A_j(x_\beta(\tau)), x_\beta(\tau)-x\rangle$$

$$+\beta^{N-i}(\tau)\langle x_\beta(\tau)-x_*, x_\beta(\tau)-x\rangle \le 0,$$

or

$$\langle A_{i+1}(x), x_\beta(\tau)-x\rangle + \sum_{j=i+2}^{N}\beta^{j-(i+1)}(\tau)\langle A_j(x), x_\beta(\tau)-x\rangle$$

$$+\beta^{N-i}(\tau)\langle x-x_*, x_\beta(\tau)-x\rangle \le 0.$$

After passing $\tau\to+\infty$, it is clear that

$$\langle A_{i+1}(x), \bar{x}-x\rangle \le 0 \ \ \forall x\in\tilde{S}_i.$$

So, $\bar{x}\in S_{i+1}$. It means that $\bar{x}\in S.$ S is a closed convex subset in *H*, because each S_j is closed convex. Hence, from (2.2) and $x_\beta(\tau)$ weak convergence to $\bar{x}$ it deduces that $\bar{x}$ is an x_*-minimal norm solution of *S*. This element is unique. Consequently, all sequence $\{x_\alpha(\tau)\}$ weak convergence to $\bar{x}$. Again, from (2.2) we have

$$\|\bar{x}-x_*\| \le \|x-x_*\| \ \ \forall x\in S.$$

Since H is a Hilbert space, then $\lim_{\tau\to+\infty} x_\alpha(\tau)=\tilde{x}.$ Theorem is proved.

Remark. It is clear that if x_α is converges weakly to $\tilde{x}\in S,$ i.e., $S\ne\varnothing.$

Now, consider the differential equation

$$\frac{dy(t,\tau)}{dt}$$

$$+\gamma(t)\left[\sum_{j=0}^{N}\alpha^j(\tau)A_j\big(y(t,\tau)\big)+\alpha^{N+1}(\tau)\big(y(t,\tau)-x_*\big)\right]=\theta,$$

$$y(t_0,\tau)=u_0, \qquad (2.3)$$

for each fixed $\tau\ge t_0$.

Theorem 2.2. *Assume that the following conditions hold:*

(*i*) *The problems* (1.5) *and* (2.3) *possess solutions in the class* $C^1[t_0,+\infty)$ *for any* $u_0 \in H$ *with* $\|u(t)\| \le d_1,\, d_1 > 0,\, t \ge t_0$;

(*ii*) *The fuctions* $\alpha(t), h(t)$ *and* $\gamma(t)$ *satisfy the condition* (1.6);

(*iii*) A_j *are bounded.*

Then, $\lim_{\tau\to+\infty} u(\tau) = \tilde{x}.$

Proof. The proof is done on the base of the techniques in [11]. For the value

$$r(t,\tau) := \|y(t,\tau) - x_\alpha(\tau)\|^2,$$

we have [2]

$$\frac{dr(t,\tau)}{dt} = 2\|y(t,\tau)-x_\alpha(\tau)\|\frac{d}{dt}\|y(t,\tau)-x_\alpha(\tau)\|$$
$$= 2\left\langle \frac{d}{dt}\big(y(t,\tau)-x_\alpha(\tau)\big), y(t,\tau)-x_\alpha(\tau)\right\rangle.$$

From (2.1) and (2.3) it follows

$$\left\langle \frac{d}{dt}\big(y(t,\tau)-x_\alpha(\tau)\big), y(t,\tau)-x_\alpha(\tau)\right\rangle$$
$$+\gamma(t)\left[\sum_{j=0}^{N}\alpha^j(\tau)\left\langle A_j\big(y(t,\tau)\big)-A_j\big(x_\alpha(\tau)\big), y(t,\tau)-x_\alpha(\tau)\right\rangle\right.$$
$$\left.+\alpha^{N+1}(\tau)\left\langle y(t,\tau)-x_\alpha(\tau), y(t,\tau)-x_\alpha(\tau)\right\rangle\right] = 0.$$

As A_j are monotone, then $r(t,\tau)$ is the solution of the following inequality [10]

$$\frac{dr(t,\tau)}{dt} + 2\gamma(t)\alpha^{N+1}(\tau) r(t,\tau) \le 0.$$

Hence,

$$r(t,\tau) \le r(t_0,\tau)\exp\left[-2\alpha^{N+1}(\tau)\int_{t_0}^{t}\gamma(t)dt\right],$$

with

$$r(t_0,\tau) = \|y(t_0,\tau)-x_\alpha(\tau)\|^2 \le \big(\|u_0\| + \|x_\alpha(\tau)\|\big)^2$$
$$\le \big(\|u_0\| + 2\|x_*\| + \|x_0\|\big)^2.$$

Consequently,

$$r(\tau,\tau) \le r(t_0,\tau)\exp\left[-2\alpha^{N+1}(\tau)\int_{t_0}^{\tau}\gamma(t)dt\right].$$

On the base of the properties of $\gamma(t)$ we have $\int_{t_0}^{+\infty}\gamma(t)dt = +\infty.$ Using (1.6) and the Lopital's rule we obtain

$$\lim_{\tau\to\infty}\alpha^{N+1}(\tau)\int_{t_0}^{\tau}\gamma(t)dt = \lim_{\tau\to\infty}\frac{\gamma(\tau)\alpha^{N+2}(\tau)}{\alpha'(\tau)} = +\infty.$$

Therefore, $\lim_{\tau\to\infty} t(\tau,\tau) = 0$ and $\|y(t,\tau)\| \le d_2$, $\forall t \ge t_0$, where d_2 is some positive constant.

Now, consider the value

$$R(t,\tau) := \|u(t) - y(t,\tau)\|^2 \quad \forall t,\tau \ge t_0.$$

From (1.5) and (2.3) it implies that

$$\left\langle \frac{d}{dt}\big(u(t)-y(t,\tau)\big), u(t)-y(t,\tau)\right\rangle$$
$$+\gamma(t)\left[\left\langle \sum_{j=0}^{N}\alpha^j(t)A_j^{h(t)}\big(u(t)\big) - \sum_{j=0}^{N}\alpha^j(\tau)A_j\big(y(t,\tau)\big), u(t)-y(t,\tau)\right\rangle\right.$$
$$\left.+\left\langle \alpha^{N+1}(t)\big(u(t)-x_*\big) - \alpha^{N+1}(\tau)\big(y(t,\tau)-x_*\big), u(t)-y(t,\tau)\right\rangle\right] = 0.$$

Here,

$$\left\langle \sum_{j=0}^{N}\alpha^j(t)A_j^{h(t)}\big(u(t)\big) - \sum_{j=0}^{N}\alpha^j(\tau)A_j\big(y(t,\tau)\big), u(t)-y(t,\tau)\right\rangle$$
$$= \sum_{j=0}^{N}\alpha^j(t)\left\langle A_j^{h(t)}\big(u(t)\big) - A_j^{h(t)}\big(y(t,\tau)\big), u(t)-y(t,\tau)\right\rangle$$
$$+\sum_{j=0}^{N}\alpha^j(t)\left\langle A_j^{h(t)}\big(y(t,\tau)\big) - A_j\big(y(t,\tau)\big), u(t)-y(t,\tau)\right\rangle$$
$$+\sum_{j=0}^{N}\left[\alpha^j(t)\text{-}\alpha^{\mu_j}(\tau)\right]\left\langle A_j\big(y(t,\tau)\big), u(t)-y(t,\tau)\right\rangle.$$

On the other hand,

$$\left\langle \alpha^{N+1}(t)\big(u(t)-x_*\big) - \alpha^{N+1}(\tau)\big(y(t,\tau)-x_*\big), u(t)-y(t,\tau)\right\rangle$$
$$= \alpha^{N+1}(t)\|u(t)-y(t,\tau)\|^2$$
$$+\big(\alpha^{N+1}(t)-\alpha^{N+1}(\tau)\big)\left\langle y(t,\tau)-x_*, u(t)-y(t,\tau)\right\rangle.$$

Since $\|u(t)\|, A_j$ are bounded, it deduces the following inequality

$$\frac{dR(t,\tau)}{dt} \le 2\gamma(t)\Big[h(t)d_3(d_1+d_2)(N+1)$$
$$+\big(d_2 + \|x_*\|\big)(d_1+d_2)\left|\alpha^{N+1}(t)-\alpha^{N+1}(\tau)\right|$$
$$+d_4(d_1+d_2)\sum_{j=0}^{N}\left|\alpha^j(t)-\alpha^j(\tau)\right|\Big] - 2\tilde{\alpha}(t)R(t,\tau),$$

$R(t_0,\tau) = \|u(t_0) - y(t_0,\tau)\| = \|u_0 - u_0\| = 0,$

$\tilde{\alpha}(t) = \gamma(t)\,\alpha^{N+1}(t),$

where $d_3 \ge g(\|u(t)\|),\ d_4 \ge \max_j \|A_j(g(t,\tau))\|.$

Hence,

$$R(t,\tau) \le M_1 \int_{t_0}^{t} \gamma(s)\Big[h(s) + \sum_{j=0}^{N+1} |\alpha^j(s) - \alpha^j(\tau)|\Big] exp\left(-\int_s^t \tilde{\alpha}(\lambda)d\lambda\right) ds,$$

where M_1 is some positive constant.

Using the equality

$$a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + ... + b^{k-1}\right)$$

and the properties of $\alpha(t)$, we have

$$R(t,\tau) \le M \int_{t_0}^{t} \gamma(s)\big[h(s) + |\alpha(s) - \alpha(\tau)|\big] exp\left(-\int_s^t \tilde{\alpha}(\lambda)d\lambda\right) ds,$$

where M is some positive constant.

From [11] we have

$$R(t,\tau) \le R_1(\tau) + R_2(\tau)$$

$$R_1(\tau) = M \int_{t_0}^{\tau} \gamma(t)h(t)\xi(t)dt / \xi(\tau)$$

$$R_2(\tau) = M \int_{t_0}^{\tau} \gamma(t)\alpha'(t)(t-\tau)\xi(t)dt / \xi(\tau),$$

$$\xi(s) = exp\left(\int_{t_0}^{s} \tilde{\alpha}(t)dt\right).$$

Therefore, $\lim_{\tau\to+\infty} R_1(\tau) = \lim_{\tau\to+\infty} R_2(\tau) = 0.$ Since

$\|\tilde{x} - u(\tau)\| \le \|\tilde{x} - x_\alpha(\tau)\| + \|x_\alpha(\tau) - y(\tau,\tau)\| + \|y(\tau,\tau) - u(\tau)\|$, then $\lim_{\tau\to+\infty} u(\tau) = \tilde{x}.$

Theorem is proved.

Remark 2.

a. The solution existences for (1.5) and (2.3) are studied in [6, 7, 8].

b. The functions $\alpha(t), h(t), \gamma(t)$ satisfying the above conditions are $h(t) = 1/t^h$, $\alpha(t) = 1/t^\alpha, \gamma(t) = 1/t^\gamma$ with $h > N\alpha$, $0 < (N+1)\alpha < 1$ and $\gamma \ge 0.$

3. Application

Give a finite family of convex functions $f_j,\ j = 0,1,...,N,$ find an $x_0 \in H$ such that

$$f_j(x_0) \le 0,\ j = 0,1,...,N.$$

Denote by

$$C_j = \{x : f_j(x) \le 0\},\ j = 0,1,...,N.$$

Then, C_j are closed convex. The problem of finding $x_0 \in \bigcap_{j=0}^{N} C_j$ is the convex feasibility one. It is intensively studied for the last time [9,10], and can be rewriten in the form of unconstrained vector convex optimization as follows. Define

$$\varphi_j(x) = \max\{0, f_j(x)\}.$$

Then C_j is coincided with the set S_j.

The problem of common fixed point is formulated as follows. Find $x_0 \in C = \bigcap_{j=0}^{N} C_j$, where $C_j = F(T_j),\ j = 0,1,...,N,$ $F(T_j)$ is the fixed point set of the nonexpensive operator T_j. It is intensively studied in recent under condition (13), (14)

$$C = F(T_N T_{N-1} ... T_0) = F(T_{N-1} ... T_0 T_N) = ... = F(T_0 T_1 ... T_N).$$

This condition can be replaced by the potential property of T_j, i.e., there exists a functional $f_j(x)$ such that $f_j'(x) = T_j(x)$ for each *j*. Then, $\varphi_j(x) = \|x\|^2/2 - f_j(x)$ is convex, since its derivative $I - T_j$ are monotone. Moreover, $S_j = C_j$, and the presented method in this paper can be applied to solve the problems.

4. Conclusion

In this paper, we proposed a continuous

method of regularization for a common minimum point of a finite system of Gâteaux differentiable weakly lower semi-continuous and properly convex functionals on real Hilbert spaces. An application to illustrate the performance of our theoretical results is also given.

REFERENCES

[1]. M. M. Vainberg, Variational method and method of monotone operators, *Moscow: Mir,* 1972.

[2]. Ng. Buong, Regularization for unconstrained vector optimization of convex functionals in Banach spaces, *Zh. Vychisl. Math. I Mat. Fiziki*, **vol. 46**, pp. 372-378, 2006.

[3]. Ng. Buong, Ph.Th.Th. Hoai and Kh.Th. Binh, Iterative regularization methods for the multiple-sets split feasibility problem in Hilbert spaces, *Acta Applicandae Mathematica*, **vol. 165**, pp. 183-197, 2020.

[4]. V. Barbu, Nonlinear semigrups and differential equations in Banach spaces, *Noordhoff Intern. Publishing Leyden Netherlands: Acad. Bucuresti, Romania*, 1976.

[5]. R.T. Rockafellar, Monotone operators and proximal point algorithm, *SIAM Journal on control and Optim*, **vol. 14**, pp. 877-897, 1976.

[6]. O. Güler , On the convergence of the proximal point algorithm for convex minimization, *SIAM Journal on control and Optim*, v**ol. 29**, pp. 403-419, 2000.

[7]. I. P. Ryazantseva, Regularization proximal algorithm for nonlinear equations of monotone type, *Zh. Vychisl. Math. i Mat. Fiziki*, **vol. 42**, pp. 1295-1303, 2002.

[8]. A. Maudafi, Second order differential proximal methods for equilibrium problems, *J. of Inequalities in Pure and Applied Math*, **vol. 4**, 2003.

[9]. H.H. Bauschke, J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Reviews*, **vol. 38**, pp. 367-426, 1996.

[10]. H.H. Bauschke, S. G. Kruk, Reflection-projection method for convex feasibility problems with an obtuse cone, *J. Optim. Theory and Appl*, **vol. 120**, pp. 503-531, 2004.

[11]. P. L. Combettes, Hibertian convex feasibility problem: convergence of projection methods, *Applied Math. And Optimization*, **vol. 35**, pp. 311-330, 1997.

[12]. F. Deutsch, I. Yamada, Minimizing certain convex finctions over the intersection of the fixed point sets for nonexpensive mappings, *Numerical Func. Anal. And Optim*, **vol. 19**, pp.33-56, 1998.

[13]. H.K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory and Appl*, **vol. 116**, pp. 659-678, 2003.

[14]. J. G. O'Hara, P. Pillay, and H.K. Xu, Iterative approach to finding nearset common fixed points of nonexpansive mappings in Hilbert spaces, *Nonlinear Anal*, **vol. 54**, pp. 1417-1426, 2003.

[15]. W. Takahashi, T. Tamura, and M. Toyoda, Approximatipon of common fixed points of a family of finite nonexpansive mappings in Hilbert spaces, *Sci. Math. Jpn*, **vol. 56**, pp. 457-480, 2002.

[16]. J. S. Jong, Y. J. Cho, and R. P. Agarwal, Iterative schemes with some control conditions for a family of nonexpansive mappings in Banach spaces, *Fixed Point Theory and Appl*, **vol. 2**, pp. 125-135, 2005.

[17]. C. E. Chidume, Convergence theorems for a common fixed point of a finite family of nonexpansive mappings, *Fixed Point Theory and Appl*, **vol. 2**, pp. 233-241, 2005.