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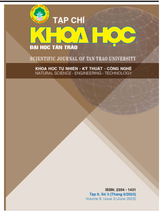

A NEW PROJECTION ALGORITHM FOR SOLVING THE SPLIT VARIATIONAL INEQUALITY PROBLEM IN HILBERT SPACES

Truong Dang Thang[1,*], *Vu Thi Thu Loan*[2]

[1] *School of Applied Mathematics and Informatics, Hanoi University of Science and Technology*

[2] *University of Agriculture and Forestry, Thai Nguyen University*

**Email address: thang.tdk64@gmail.com*

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Abstract:

This paper proposes a new algorithm for solving the split variational inequality problem in Hilbert spaces. In order to solve this problem, we propose a new algorithm and establish a strong convergence theorem for it. Compared with the work by Censor et al. (Numer. Algor., 59:301-323, 2012), the new algorithm gives strong convergence results. It shows that the iterative method converges strongly under weaker assumptions than the ones used recently. Some numerical examples are also given to illustrate the convergence analysis of the considered algorithm.

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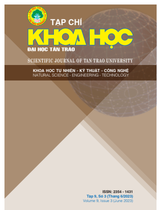

MỘT THUẬT TOÁN CHIẾU MỚI GIẢI BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN TÁCH TRONG KHÔNG GIAN HILBERT

Trương Đăng Thắng[1,*], *Vũ Thị Thu Loan*[2]

[1] *Viện Toán ứng dụng và Tin học, Đại học Bách khoa Hà Nội*

[2] *Trường Đại học Nông Lâm, Đại học Thái Nguyên*

**Email address: thang.tdk64@gmail.com*

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Từ khóa:

Bài toán bất đẳng thức biến phân tách, bài toán chấp nhận tách, không gian Hilbert, phép chiếu mê-tric.

Tóm tắt:

Bài báo đề xuất một thuật toán mới giải bài toán bất đẳng thức biến phân tách trong không gian Hilbert. Để giải bài toán này, chúng tôi đề xuất một thuật toán mới và thiết lập sự hội tụ mạnh. So sánh với thuật toán của Censor và các cộng sự (Numer. Algor., 59:301-323, 2012), thuật toán mới này cho sự hội tụ mạnh. So với một số kết quả gần đây, thuật toán của chúng tôi cho sự hội tụ mạnh dưới các điều kiện yếu hơn. Một số ví dụ cũng được đưa ra để minh họa cho sự hội tụ giải tích của thuật toán đề xuất.

1 INTRODUCTION

The split variational inequality problem (SVIP), which was introduced first by Censor et al. [1]

$$\text{find } u^* \in \Omega := S_{(A,C)} \bigcap F^{-1}\big(S_{(B,Q)}\big), \quad \text{(SVIP)}$$

where $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ are nonempty closed convex subsets, $F : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear mapping. $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ are single–valued operators, $S_{(A,C)}$ and $S_{(B,Q)}$ denote as the set of all solutions of the variational inequality problems

$$\langle Au^*, u - u^*\rangle \geq 0 \quad \forall u \in C \qquad \text{(VIP}(A, C))$$

and $\langle Bu^*, u - u^*\rangle \geq 0, \ \forall u \in Q$, respectively.

In this paper, using the viscosity approximation method [2], as well as a modification of the CQ method [3] we propose a new convergence strongly algorithm for solving the (SVIP).

2 PRELIMINARIES

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle ., .\rangle$ and norm $\|.\|$ and C be a nonempty, closed, and convex subset of $\mathcal{H}$. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicates that the sequence $\{x^k\}$ converges strongly to x. It is known that in a Hilbert space $\mathcal{H}$,

$$\begin{aligned} 2\langle x, y\rangle &= \|x+y\|^2 - \|x\|^2 - \|y\|^2 \\ &= \|x\|^2 + \|y\|^2 - \|x-y\|^2, \end{aligned} \quad (2.1)$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \quad (2.2)$$

for all $x, y \in \mathcal{H}$ and $\lambda \in [0,1]$ (see, for example [4, Lemma 2.13], [5]). For every point $x \in \mathcal{H}$ there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies $\|x - P_C x\| \leq \|x - u\|$ for all $u \in C$. The mapping $P_C : \mathcal{H} \to C$ is called the metric projection of $\mathcal{H}$ onto C.

Lemma 2.1 (see, [6]). *For given $x \in \mathcal{H}$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, z - y\rangle \leq 0$ for all $z \in C$.*

Definition 2.1. An operator $T : \mathcal{H} \to \mathcal{H}$ is called a contraction operator with the contraction coefficient $\tau \in [0,1)$ if $\|Tx - Ty\| \leq \tau\|x - y\|$ for all $x, y \in \mathcal{H}$.

It is easy to see that, if T is a contraction operator, then $P_C T$ is a contraction operator too. If $\tau \geq 0$ we have τ-Lipschitz continuous operator.

Definition 2.2. An operator $A : \mathcal{H} \to \mathcal{H}$ is called an η-inverse strongly monotone operator with constant $\eta > 0$ if $\langle Ax - Ay, x - y\rangle \geq \eta\|Ax - Ay\|^2$ for all $x, y \in \mathcal{H}$.

It is easy to see that, if A is an η-inverse strongly monotone operator, then $I^{\mathcal{H}} - \lambda A$ is a nonexpansive mapping for $\lambda \in (0, 2\eta]$, where $I^{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

Lemma 2.2 (see [1]). *Let $A : C \to \mathcal{H}$ be η-inverse strongly monotone on C and $\lambda > 0$ be a constant satisfying $0 < \lambda \leq 2\eta$. Define the mapping $T : C \to C$ by taking*

$$Tx = P_C\big(I^{\mathcal{H}} - \lambda A\big)x \quad \textit{for all} \quad x \in C. \quad (2.3)$$

Then T is nonexpansive on C, and $\mathrm{Fix}(T) = S_{(A,C)}$, where $\mathrm{Fix}(T) := \{x \in C \mid Tx = x\}$ is the set of fixed points of T.

Lemma 2.3 (see, [6]). *Assume that T be a nonexpansive mapping of a closed and convex subset C of a Hilbert space $\mathcal{H}$ into $\mathcal{H}$. Then the mapping $I^{\mathcal{H}} - T$ is demiclosed on C; that is, whenever $\{x^k\}$ is a sequence in C which weakly converges to some point $u^* \in C$ and the sequence $\{(I^{\mathcal{H}} - T)x^k\}$ strongly converges to some y, it follows that $(I^{\mathcal{H}} - T)u^* = y$.*

From Lemma 2.3, if $x^k \rightharpoonup u^*$ and $(I^{\mathcal{H}} - T)x^k \to 0$, then $u^* \in \mathrm{Fix}(T)$.

Lemma 2.4 (Maingé, [7]). *Let $\{s_k\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{s_{k_n}\}$ such that $s_{k_n} \leq s_{k_n+1}$ for all $n \geq 0$. Define an integer sequence by $\nu(k) := \max\{k_0 \leq n \leq k \mid s_n < s_{n+1}\}$, $k \geq k_0$. Then $\nu(k) \to \infty$ as $k \to \infty$ and for all $k \geq k_0$, we have $\max\{s_{\nu(k)}, s_k\} \leq s_{\nu(k)+1}$.*

Lemma 2.5 (see, [8]). *Let $\{s_k\}$ be a sequence of nonnegative numbers satisfying the condition $s_{k+1} \leq (1 - b_k)s_k + b_k c_k$, $k \geq 0$, where $\{b_k\}$ and $\{c_k\}$ are sequences of real numbers such that*

(i) $\{b_k\} \subset (0,1)$ *for all* $k \geq 0$ *and* $\sum_{k=1}^{\infty} b_k = \infty$,

(ii) $\limsup_{k\to\infty} c_k \leq 0$.

Then, $\lim_{k\to\infty} s_k = 0$.

3 MAIN RESULTS

We consider the (SVIP) under the following conditions.

Assumption 3.1.

(A1) $A : \mathcal{H}_1 \to \mathcal{H}_1$ is an η_A-inverse strongly monotone on $\mathcal{H}_1$.

(A2) $B : \mathcal{H}_2 \to \mathcal{H}_2$ is an η_B-inverse strongly monotone on $\mathcal{H}_2$.

(A3) $F : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

(A4) $T : \mathcal{H}_1 \to \mathcal{H}_1$ is a contraction mapping with the contraction coefficient $\tau \in [0,1)$.

(A4) The solution set Ω of the (SVIP) is nonempty.

We also consider some conditions.

$$\{\alpha_k\} \subset (0,1) \quad \text{for all } k \geq 0,$$

$$\lim_{k\to\infty} \alpha_k = 0, \ \sum_{k=0}^{\infty} \alpha_k = \infty; \quad (\alpha)$$

$$0 < \lambda \leq 2\eta; \ \eta = \min\{\eta_A, \eta_B\}; \quad (\lambda)$$

$$0 < \gamma < \frac{1}{\|F\|^2}. \quad (\gamma)$$

We present a algorithm for solving the (SVIP). This is our new algorithm.

Algorithm 3

Step 0. Select the initial point $x^0 \in \mathcal{H}_1$ and the sequence $\{\beta_k\} \subset [c,d] \subset (0,1)$ $\forall k \geq 0$, the sequences $\{\alpha_k\}$, λ, and γ such that the conditions (α), (λ), and (γ) are satisfied. Set $k := 0$.

Step 1. Compute $u^k = \beta_k x^k + (1-\beta_k)P_C^{\mathcal{H}_1}(x^k - \lambda A x^k)$.

Step 2. Compute $v^k = P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k))$.

Step 3. Compute $w^k = u^k + \gamma F^*(v^k - Fu^k)$.

Step 4. Compute $x^{k+1} = \alpha_k T(x^k) + (1-\alpha_k)w^k$.

Step 5. Set $k := k+1$ and go to **Step 1**.

Theorem 3.1. *Suppose that all conditions in Assumption 3.1 are satisfied. Then the sequence $\{x^k\}$ generated by Algorithm 3 converges strongly to the unique solution $u^* \in \Omega$ of the* VIP$(I^{\mathcal{H}_1} - T, \Omega)$.

Proof. Since T is a contraction mapping, $P_\Omega T$ is a contraction too. By Banach contraction mapping principle, there exists a unique point $u^* \in \Omega$ such that $P_\Omega T u^* = u^*$. By Lemma 2.1, we obtain u^* is the unique solution to the VIP$(I^{\mathcal{H}_1} - T, \Omega)$.

1. Claim the sequence $\{x^k\}$ is well defined.

Indeed, let $u \in \Omega$. Since $u \in \Omega$, $u \in S_{(A,C)}$. It follows from (λ) and Lemma 2.2 that $u = P_C^{\mathcal{H}_1}(I^{\mathcal{H}_1} - \lambda A)u$. From Step 1 in Algorithm 3, the nonexpansive property of $P_C^{\mathcal{H}_1}(I^{\mathcal{H}_1} - \lambda A)$, $\{\beta_k\} \subset [c,d] \subset (0,1)$ $\forall k \geq 0$, and (2.2), we have that

$$\begin{aligned}
\|u^k - u\|^2 &= \Big\|\beta_k(x^k - u) + (1-\beta_k)\left[P_C^{\mathcal{H}_1}(x^k - \lambda A x^x) - u\right]\Big\|^2 \\
&= \Big\|\beta_k(x^k - u) + (1-\beta_k)\left[P_C^{\mathcal{H}_1}(x^k - \lambda A x^k) - P_C^{\mathcal{H}_1}(u - \lambda A u)\right]\Big\|^2 \\
&= \beta_k\|x^k - u\|^2 + (1-\beta_k)\|x^k - u\|^2 - \beta_k(1-\beta_k)\|x^k - P_C^{\mathcal{H}_1}(x^k - \lambda A x^k)\|^2\Big] \\
&= \|x^k - u\|^2 - \beta_k(1-\beta_k)\|x^k - P_C^{\mathcal{H}_1}(x^k - \lambda A x^k)\|^2 \qquad (3.1) \\
&\leq \|x^k - u\|^2. \qquad (3.2)
\end{aligned}$$

It follows from Step 3 in Algorithm 3, the property of adjoint operator F^*, and (2.1) that

$$\begin{aligned}
\|w^k - u\|^2 &= \left\|u^k + \gamma F^*(v^k - Fu^k) - u\right\|^2 \\
&= \|u^k - u\|^2 + \gamma^2\left\|F^*(v^k - Fu^k)\right\|^2 + 2\gamma\langle u^k - u, F^*(v^k - Fu^k)\rangle \\
&= \|u^k - u\|^2 + \gamma^2\|F\|^2\|v^k - Fu^k\|^2 + 2\gamma\langle Fu^k - Fu, v^k - Fu^k\rangle. \qquad (3.3)
\end{aligned}$$

Using the convexity of $\|\cdot\|^2$ and Step 2 in Algorithm 3, we have

$$\|v^k - Fu^k\|^2 = \left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu^k\right\|^2. \qquad (3.4)$$

Since $u \in \Omega$, $Fu \in S_{(B,Q)}$. It follows from Lemma 2.2 that $Fu = P_Q^{\mathcal{H}_2}(I^{\mathcal{H}_2} - \lambda B)Fu$. From Step 2 in Algorithm 3, the nonexpansive property of $P_Q^{\mathcal{H}_2}(I^{\mathcal{H}_2} - \lambda B)$, we have

$$\begin{aligned}
\langle Fu^k - Fu, v^k - Fu^k\rangle &= \langle Fu^k - Fu, P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu^k\rangle \\
&= \frac{1}{2}\Big(\left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu\right\|^2 - \left\|Fu^k - Fu\right\|^2 - \left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu^k\right\|^2\Big) \\
&= \frac{1}{2}\Big(\left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - P_Q^{\mathcal{H}_2}(Fu - \lambda B(Fu))\right\|^2 - \left\|Fu^k - Fu\right\|^2 - \left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu^k\right\|^2\Big) \\
&\leq -\frac{1}{2}\left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu^k\right\|^2. \qquad (3.5)
\end{aligned}$$

It folows from (3.3)–(3.5) and (γ) that

$$\begin{aligned}
\|w^k - u\|^2 &\leq \|u^k - u\|^2 - \gamma(1 - \gamma\|F\|^2)\left\|P_Q^{\mathcal{H}_2}(Fu^k - \lambda B(Fu^k)) - Fu^k\right\|^2 \qquad (3.6) \\
&\leq \|u^k - u\|^2. \qquad (3.7)
\end{aligned}$$

It follows from the convexity of the norm function $\|.\|$ on $\mathcal{H}_1$, the contraction property of T with the contraction coefficient $\tau \in [0,1)$, (3.2), (3.7), the condition (α), and Step 4 in Algorithm 3 that

$$\begin{aligned}\|x^{k+1}-u\| &= \big\|\alpha_k(Tx^k-u)+(1-\alpha_k)(w^k-u)\big\| \\ &\le \alpha_k\big(\|Tx^k-Tu\|+\|Tu-u\|\big) \\ &\quad +(1-\alpha_k)\|w^k-u\| \\ &\le \tau\alpha_k\|x^k-u\|+\alpha_k\|Tu-u\| \\ &\quad +(1-\alpha_k)\|x^k-u\| \\ &= \big[1-(1-\tau)\alpha_k\big]\|x^k-u\| \\ &\quad +(1-\tau)\alpha_k\frac{\|Tu-u\|}{1-\tau} \\ &\le \max\Big\{\|x^k-u\|,\frac{\|Tu-u\|}{1-\tau}\Big\} \\ &\ \ \vdots \\ &\le \max\Big\{\|x^0-u\|,\frac{\|Tu-u\|}{1-\tau}\Big\}.\end{aligned}$$

This implies that the sequence $\{x^k\}$ is bounded. Since P_C and P_Q are nonexpansive mappings and F is the bounded linear operator, we also have the sequences $\{u^k\}$, $\{v^k\}$, and $\{w^k\}$ are bounded.

2. For any $u \in \Omega$, the following inequality holds:

$$s_{k+1} \le \big[1-(1-\tau)\alpha_k\big]s_k+\alpha_k e_k, \qquad (3.8)$$

where $s_k := \|x^k-u\|^2$ and $e_k := 2\langle Tu-u, x^{k+1}-u\rangle$.

Indeed, from the convexity of $\|.\|^2$, Step 4 in Algorithm 3, (3.1), (3.6), and the condition (α), we get

$$\begin{aligned}&\|x^{k+1}-u\|^2 = \big\|\alpha_k(Tx^k-u)+(1-\alpha_k)(w^k-u)\big\|^2 \\ &\le \alpha_k\big\|Tx^k-u\big\|^2+(1-\alpha_k)\|w^k-u\|^2 \\ &\le \alpha_k\big\|Tx^k-u\big\|^2+\|u^k-u\|^2 \\ &\ -\gamma(1-\gamma\|F\|^2)\Big\|P_Q^{\mathcal{H}_2}(Fu^k-\lambda B(Fu^k))-Fu^k\Big\|^2 \\ &\le \alpha_k\big\|Tx^k-u\big\|^2+\|x^k-u\|^2-\gamma(1-\gamma\|F\|^2) \\ &\quad \Big\|P_Q^{\mathcal{H}_2}(Fu^k-\lambda B(Fu^k))-Fu^k\Big\|^2 \\ &\quad -\beta_k(1-\beta_k)\Big\|x^k-P_C^{\mathcal{H}_1}(x^k-\lambda A(x^k))\Big\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\gamma(1-\gamma\|F\|^2)\Big\|P_Q^{\mathcal{H}_2}(Fu^k-\lambda B(Fu^k))-Fu^k\Big\|^2 \\ &\quad +\beta_k(1-\beta_k)\Big\|x^k-P_C^{\mathcal{H}_1}(x^k-\lambda A(x^k))\Big\|^2 \\ &\le \Big(\|x^k-u\|^2-\|x^{k+1}-u\|^2\Big)+\alpha_k\big\|Tx^k-u\big\|^2.\end{aligned} \qquad (3.9)$$

Next, from Step 4 in Algorithm 3 and the contraction property of T with the contraction coefficient $\tau \in [0,1)$, we have that

$$\begin{aligned}\|x^{k+1}-u\|^2 &= \langle\alpha_k(Tx^k-u)+(1-\alpha_k)(w^k-u), \\ &\quad x^{k+1}-u\rangle \\ &= (1-\alpha_k)\langle w^k-u, x^{k+1}-u\rangle \\ &\quad +\alpha_k\langle Tx^k-u, x^{k+1}-u\rangle \\ &\le \frac{1-\alpha_k}{2}\Big(\|w^k-u\|^2+\|x^{k+1}-u\|^2\Big) \\ &\quad +\alpha_k\langle Tx^k-Tu, x^{k+1}-u\rangle \\ &\quad +\alpha_k\langle Tu-u, x^{k+1}-u\rangle \\ &\le \frac{1-\alpha_k}{2}\Big(\|w^k-u\|^2+\|x^{k+1}-u\|^2\Big) \\ &\quad +\frac{\alpha_k}{2}\Big(\tau\|x^k-u\|^2+\|x^{k+1}-u\|^2\Big) \\ &\quad +\alpha_k\langle Tu-u, x^{k+1}-u\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}&\|x^{k+1}-u\|^2 \le (1-\alpha_k)\|w^k-u\|^2 \\ &+\alpha_k\tau\|x^k-u\|^2+2\alpha_k\langle Tu-u, x^{k+1}-u\rangle.\end{aligned} \qquad (3.10)$$

From (3.2), (3.7), and (3.10), we obtain

$$\begin{aligned}&\|x^{k+1}-u\|^2 \le \big[1-(1-\tau)\alpha_k\big]\|x^k-u\|^2 \\ &\quad +2\alpha_k\langle Tu-u, x^{k+1}-u\rangle.\end{aligned} \qquad (3.11)$$

Put $s_k := \|x^k-u\|^2$ and $e_k := 2\langle Tu-u, x^{k+1}-u\rangle$, then the inequality (3.11) can be rewritten as (3.8).

3. We will show $\lim_{n\to\infty}\|x^k-u^*\| = 0$, where $u^* = P_\Omega Tu^*$.

We consider two possible cases.

Case 1. There exists an integer $k_0 \ge 0$ such that $\|x^{k+1}-u^*\| \le \|x^k-u^*\|$ for all $k \ge k_0$. Then, $\lim_{k\to\infty}\|x^k-u^*\|$ exists. Since the sequence $\{x^k\}$ is bounded, the sequence $\{Tx^k\}$ is also bounded. From the boundedness of the sequence $\{Tx^k\}$, (α), (λ), and (γ), it follows from (3.9) that

$$\lim_{k\to\infty}\big\|\big[I^{\mathcal{H}_1}-P_C^{\mathcal{H}_1}\big(I^{\mathcal{H}_1}-\lambda A\big)\big]x^k\big\| = 0 \qquad (3.12)$$

and

$$\lim_{k\to\infty}\big\|\big[I^{\mathcal{H}_2}-P_Q^{\mathcal{H}_2}\big(I^{\mathcal{H}_2}-\lambda B\big)\big]Fu^k\big\| = 0. \qquad (3.13)$$

From the fact that (3.13) and (3.4), we get

$$\lim_{k\to\infty}\|v^k-Fu^k\| = 0. \qquad (3.14)$$

From Step 3 in Algorithm 3, the property of adjoint operator F^*, and (3.14), we obtain

$$\lim_{k\to\infty}\|w^k-u^k\| = \gamma\lim_{k\to\infty}\|F^*(v^k-Fu^k)\| = 0. \qquad (3.15)$$

From Step 1 in Algorithm 3 and (3.13), we get

$$\lim_{k\to\infty} \|x^k - u^k\| = \lim_{k\to\infty} (1-\beta_k)\|x^k - P_C^{\mathcal{H}_1}(I^{\mathcal{H}_1} - \lambda A)x^k\| = 0. \quad (3.16)$$

It follows from (3.16) and (3.15) that

$$\lim_{k\to\infty} \|x^k - w^k\| = 0. \quad (3.17)$$

Using the boundedness of $\{w^k\}$ and $\{Tx^k\}$, Step 4 in Algorithm 3, and the condition (α), we also have $\lim_{k\to\infty} \|x^{k+1} - w^k\| = \lim_{k\to\infty} \alpha_k\|Tx^k - w^k\| = 0$. When combined with (3.17), this implies that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \quad (3.18)$$

Now we show that $\limsup_{k\to\infty}\langle Tu^* - u^*, x^{k+1} - u^*\rangle \le 0$. Indeed, suppose that $\{x^{k_n}\}$ is a subsequence of $\{x^k\}$ such that

$$\limsup_{k\to\infty}\langle Tu^* - u^*, x^k - u^*\rangle = \lim_{k_n\to\infty}\langle Tu^* - u^*, x^{k_n} - u^*\rangle. \quad (3.19)$$

Since $\{x^{k_n}\}$ is bounded, there exists a subsequence $\{x^{k_{n_l}}\}$ of $\{x^{k_n}\}$ which converges weakly to some points $u^\dagger$. Without loss of generality, we may assume that $x^{k_n} \rightharpoonup u^\dagger$. We will prove that $u^\dagger \in \Omega$. Indeed, from (3.12), Lemma 2.2 and Lemma 2.3, we obtain $u^\dagger \in S_{(A,C)}$. Moreover, since F is a bounded linear operator, $Fx^{k_n} \rightharpoonup Fu^\dagger$. Using (3.13), Lemma 2.2 and Lemma 2.3, we also obtain $Fu^\dagger \in S_{(B,C)}$. Hence, $u^\dagger \in \Omega$. So, from $u^* = P_\Omega Tu^*$, (3.19), and Lemma 2.1 we deduce that $\limsup_{k\to\infty}\langle Tu^* - u^*, x^k - u^*\rangle = \langle Tu^* - u^*, u^\dagger - u^*\rangle \le 0$, which combined with (3.18) gives

$$\limsup_{k\to\infty}\langle Tu^* - u^*, x^{k+1} - u^*\rangle \le 0. \quad (3.20)$$

Now, the inequality (3.8) with u replaced by u^*, can be rewritten in the form $s_{k+1} \le (1-b_k)s_k + b_k c_k$, where $b_k = (1-\tau)\alpha_k$ and $c_k = \frac{2}{1-\tau}\langle Tu^* - u^*, x^{k+1} - u^*\rangle$. Since the condition ($\alpha$) and $\tau \in [0,1)$, $\{b_k\} \subset (0,1)$ and $\sum_{k=1}^{\infty} b_k = \infty$. Consequently, from $\tau \in [0,1)$ and (3.20), we have that $\limsup_{k\to\infty} c_k \le 0$. Finally, by Lemma 2.5, $\lim_{k\to\infty} s_k = 0$. Hence, $\lim_{k\to\infty}\|x^k - u^*\| = 0$.

Case 2. There exists a subsequence $\{k_n\}$ of $\{k\}$ such that $\|x^{k_n} - u^*\| \le \|x^{k_n+1} - u^*\|$ for all $n \ge 0$. Hence, by Lemma 2.4, there exists an integer, non-decreasing sequence $\{\nu(k)\}$ for $k \ge k_0$ (for some k_0 large enough) such that $\nu(k) \to \infty$ as $k \to \infty$,

$$\|x^{\nu(k)} - u^*\| \le \|x^{\nu(k)+1} - u^*\| \quad \text{and}$$
$$\|x^k - u^*\| \le \|x^{\nu(k)+1} - u^*\| \quad \text{for each } k \ge 0. \quad (3.21)$$

From (3.8) with u replaced by u^* and k replaced by $\nu(k)$, we have

$$0 < \|x^{\nu(k)+1} - u^*\|^2 - \|x^{\nu(k)} - u^*\|^2 \le 2\alpha_{\nu(k)}\langle Tu^* - u^*, x^{\nu(k)+1} - u^*\rangle.$$

Since $\alpha_{\nu(k)} \to 0$ and the boundedness of $\{x^{\nu(k)}\}$, we conclude that

$$\lim_{k\to\infty}\left(\|x^{\nu(k)+1} - u^*\|^2 - \|x^{\nu(k)} - u^*\|^2\right) = 0. \quad (3.22)$$

By a similar argument to Case 1, we obtain $\lim_{k\to\infty}\left\|\left[I^{\mathcal{H}_1} - P_C^{\mathcal{H}_1}(I^{\mathcal{H}_1} - \lambda A)\right]x^{\nu(k)}\right\| = 0$ and $\lim_{k\to\infty}\left\|\left[I^{\mathcal{H}_2} - P_Q^{\mathcal{H}_2}(I^{\mathcal{H}_2} - \lambda B)\right]Fu^{\nu(k)}\right\| = 0$. Also we get

$$\|x^{\nu(k)+1} - u^*\|^2 \le \left[1 - (1-\tau)\alpha_{\nu(k)}\right]\|x^{\nu(k)} - u^*\|^2 + 2\alpha_{\nu(k)}\langle Tu^* - u^*, x^{\nu(k)+1} - u^*\rangle,$$

where $\limsup_{k\to\infty}\langle Tu^* - u^*, x^{\nu(k)+1} - u^*\rangle \le 0$. Since the first inequality in (3.21) and $\alpha_{\nu(k)} > 0$, we have that $(1-\tau)\|x^{\nu(k)} - u^*\|^2 \le 2\langle Tu^* - u^*, x^{\nu(k)+1} - u^*\rangle$. Thus, from $\limsup_{k\to\infty}\langle Tu^* - u^*, x^{\nu(k)+1} - u^*\rangle \le 0$ and $\tau \in [0,1)$, we get $\lim_{k\to\infty}\|x^{\nu(k)} - u^*\|^2 = 0$. This together with (3.22) implies that $\lim_{k\to\infty}\|x^{\nu(k)+1} - u^*\|^2 = 0$. Which together with the second inequality in (3.21) implies that $\lim_{k\to\infty}\|x^k - u^*\| = 0$.

Since T is a contraction mapping, $P_\Omega T$ is a contraction too. By Banach contraction mapping principle, there exists a unique point $u^* \in \Omega$ such that $P_\Omega Tu^* = u^*$. By Lemma 2.1, we obtain u^* is the unique solution to the VIP$(I^{\mathcal{H}_1} - T, \Omega)$. This completes the proof.

4 NUMERICAL EXPERIMENTS

We perform the iterative schemes in Python running on a laptop with Intel Core i7 8650U CPU, 16GB RAM.

Example 4.1. In this example, with the purpose of illustrating the convergence of the Algorithm 3, we will apply the method to solve (SVIP). Let $\mathcal{H}_1 = \mathbb{R}^4$ and $\mathcal{H}_2 = \mathbb{R}^5$. Operators $A : \mathbb{R}^4 \to \mathbb{R}^4$

and $B : \mathbb{R}^5 \to \mathbb{R}^5$ are defined by

$$Ax = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 7 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \text{ and}$$

$$Bx = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5$$

that are inverse strongly monotone operator with constant $\eta_A = \frac{1}{9}$ and $\eta_B = \frac{1}{7}$, respectively. Bounded linear operator $F : \mathbb{R}^4 \to \mathbb{R}^5$,

$$Fx = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 7 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4.$$

And $Tx : \mathbb{R}^4 \to \mathbb{R}^4$,

$$Tx = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0.25 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4,$$

is contractive operator with constant $\tau = \frac{1}{2}$. Let C and Q are defined by $C = \{x = (x_1, x_2, x_3, x_4) \mid 2x_1 + x_4 \leq 1\}; Q = \{y = (y_1, y_2, y_3, y_4, y_5) \mid y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 \leq 1\}$. The solutions set of (SVIP) is

$$\Omega = \Big\{ x = (-u - v, \ u, \ 0, \ v) \mid 9u^2 + v^2 \leq 1; \ 2u + v \geq -1; \ u, v \in \mathbb{R} \Big\}.$$

The unique solution of VIP $\left(I^{\mathbb{R}^4} - T, \Omega\right)$ is $x^* = \begin{pmatrix} -0.3 & 0.1 & 0 & 0.2 \end{pmatrix}^\top$.

Now, choose $\alpha_k = \frac{1}{\sqrt{k+1}}, \lambda = 0.2, \beta_k = 0.25, \gamma = 0.01$, tolerance $\varepsilon = 10^{-6}$ and initial point $x^0 = (2 \quad -1 \quad 0 \quad 5)^\top$, we get $x = (-0.2943, \ 0.1056, \ -0.0014, \ 0.2056)^\top$. This result archived within 0.208041 seconds.

Next, we used different choices of parameters. Table shown below is the performance with different λ parameter, $(0 < \lambda \leq 2\eta \approx 0.2222)$ and $\alpha_k = \frac{1}{\sqrt{k+1}}, \beta_k = 0.25, \gamma = 0.01$ with initial point $x^0 = (2 \quad -1 \quad 0 \quad 5)^\top$. Tolerance $\varepsilon = 10^{-6}$.

λ	Number of iterations	Time
0.05	13557	0.5560s
0.10	8514	0.3500s
0.15	6303	0.2649s
0.20	4963	0.2080s

Bảng 4.1: Results with different λ

Then, we changed the parameter γ with $0 < \gamma < \frac{1}{L} = \frac{1}{54} \approx 0.0185$. The other parameters stay unchanged $\lambda = 0.20, \alpha_k = \dfrac{1}{\sqrt{k+1}}, \beta_k = 0.25$ with initial point $x^0 = (2 \quad -1 \quad 0 \quad 5)^\top$. Tolerance $\varepsilon = 10^{-6}$.

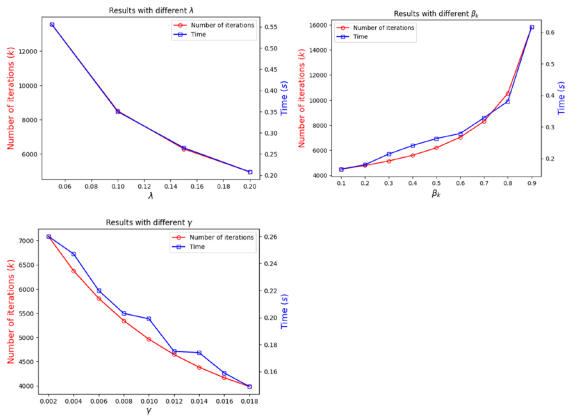
γ	Number of iterations	Time
0.002	7088	0.260000s
0.004	6378	0.247037s
0.006	5808	0.219998s
0.008	5345	0.203006s
0.010	4963	0.199033s
0.012	4647	0.174994s
0.014	4385	0.174000s
0.016	4167	0.159012s
0.018	3987	0.148996s

Bảng 4.2: Results with different γ

Following that, we changed the parameter β_k as well, with the same choice of parameters, as $\lambda = 0.20, \alpha_k = \dfrac{1}{\sqrt{k+1}}, \gamma = 0.01$ with initial point $x^0 = (2 \quad -1 \quad 0 \quad 5)^\top$. Tolerance $\varepsilon = 10^{-6}$.

β_k	Number of iterations	Time
0.1	4501	0.166038s
0.2	4793	0.180050s
0.3	5152	0.214039s
0.4	5606	0.241030s
0.5	6205	0.263004s
0.6	7040	0.278997s
0.7	8307	0.329004s
0.8	10529	0.381039s
0.9	15828	0.617035s

Bảng 4.3: Results with different β_k

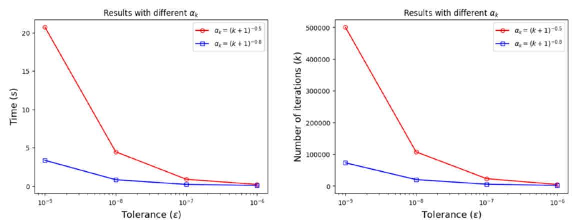


Hình 1: Results with different change in some parameters

Afterwards, we modify the parameter α_k. The table below show the results of the algorithm with $\lambda = 0.20, \beta_k = 0.25, \gamma = 0.01.$ and initial point $x^0 = (2 \quad -1 \quad 0 \quad 5)^\top$. Tolerance $\varepsilon = 10^{-6}$.

α_k	ϵ	Number of iterations (k)	Time (s)
$\alpha_k = (k+1)^{-0.5}$	10^{-6}	4963	0.208041
	10^{-7}	23133	0.882029
	10^{-8}	107595	4.463039
	10^{-9}	499903	20.689995
$\alpha_k = (k+1)^{-0.8}$	10^{-6}	1693	0.07303
	10^{-7}	5658	0.209031
	10^{-8}	20287	0.826946
	10^{-9}	72908	3.344028

Bảng 4.4: Results with different α_k



Hình 2: The behavior of the number of iterations and time when α_k changed

5 CONCLUSION

In this paper, we introduced a new algorithm (Algorithm 3) and a new strong convergence theorem for solving the (SVIP) in a real Hilbert spaces. We consider a numerical example to illustrate the effectiveness of the proposed algorithm.

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